Finite Element Formulations

for Statics and Dynamics of Plane Structures (with Matlab)

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Dedication

To our families and students…

to the future!

Preface

Finite Element Method (FEM) has become a compulsory knowledge for present day engineers as it allows (what used to be) very complex behavior of physical phenomenon to be known (approximately) and exploited. However, the teaching and learning of the subject are still difficult, as usually described by the learners. In the authors' opinion, the difficulties can be blamed on the fragmentation (of the discussions) between mathematics, engineering fundamentals and the basic concepts of numerical method. Realizing this, the authors are promoting a new approach in this book by insisting for a "close-loop" type of discussion in each topic or chapter. A topic always begins with the derivation of the differential equation/s (of the problem). It is followed by the conversion of the equation/s into matrix forms through finite element argument. A worked example is then immediately given (in a very detailed manner) before it is closed by a MATLAB source code.

This book is neither designed to be a complete book on FEM nor intended to dwell on the practice of FEM modelling (using on-shelf software). Instead, it is prepared with a specific idea in mind; the book is about easy tracing of the evolution of the finite element formulation and thus has the following features:

1. A complete loop in each formulation (from the derivation of the partial/ordinary differential equations to the discretization of the equations into matrix system to the computer programming)

2. Increasing complexity from one formulation to another (that is, from bar element to beam element to truss element to frame to free vibration and buckling problems and finally to forced vibration of the structures)

For the above reasons, this book does not have abundant worked examples but focusing on a few examples, detailing every step so as to make obvious what has been discussed in the preceding text and what awaits in succeeding source code. Also (with the specific example per chapter), the evolution and the continuity of arguments can be clearly established from one chapter to another (It is the authors' opinion that too many examples per chapter would make the relationship between examples in different chapters less obvious). Nevertheless, there are plenty of solved exercises provided.

This book has evolved from a series of lecture notes of the first author refined over the period of ten years with the co-authors. It revolves around frame structural analysis, both statics and dynamics. In Chapter 1, the book begins with the basic concepts of numerical methods before introducing the concept of Galerkin weighted residual method towards the end. Chapter 2 focuses on bar finite element. The formulation of beam element is discussed in Chapter 3. Chapter 4 discusses the concept of space orientation and the assembly of elements for plane structures (truss and frame). Chapter 5 discusses two classes of eigenvalue problems; free vibration and buckling of structures. Chapter 6 details the formulation of forced vibration of bar, beam and plane frame. In this final chapter, time discretization by finite difference method is introduced.

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October, 2018
Putrajaya, Malaysia

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1 Basic Concept of Numerical Techniques

1.1 Introduction: What is Finite Element Method?

Finite Element Method or FEM can be both "everything" and "nothing". At one end, FEM is everything when it allows engineers to get information (i.e. stresses, displacements, forces) of complex physical phenomenon for design purposes. At the other end, FEM is nothing because the information obtained is actually nothing but a solution to a partial differential equation (PDE) or ordinary differential equations (ODE). In other words, FEM is nothing but another numerical method to solve PDE or ODE.

Realizing how FEM can be "everything" is important as it can motivate the study. But realizing how FEM can be "nothing" is just as important as it can guide the proper learning of FEM that is, any discussion must begin from the first principle (i.e. PDE or ODE) if strong understanding is desired.

To note, since ODE is a special case of PDE, from now on, PDE will be quoted when references to both class of equations are made.

A formal description of FEM can be given as follows. FEM is a numerical method that approximates the solution of a PDE by breaking up the physical domain into smaller elements where adjacent elements are connected at nodes to form a mesh. Such a mesh formation process is technically termed as element assembly. In FEM, the dependent variables at nodal locations (referred as the degree of freedoms) are interpolated by shape functions. Insertion of these interpolation functions into the PDE produces a residual error function which, when forced to zero with the employment of weighted residual method, in turn, produces a matrix system. Imposition of boundary conditions can be done directly before the unknown degree of freedoms be solved.

1.2 Basic Concept of Numerical Techniques

Having said how FEM is just another numerical method, below is the list of established numerical methods.

i. Finite Element Method (FEM)
ii. Finite Difference Method (FDM)
iii. Boundary Element Method (BEM)
iv. Meshless or Meshfree Methods (Meshfree)

However, despite their variations, all the methods share similar concept that is;

> *"to convert the continuous nature of PDE (or ODE) into 'equivalent' simultaneous algebraic equations in the form of a matrix system".*

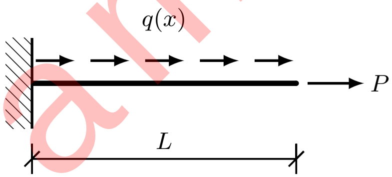

Figure 1.1: Bar element / structure.

In elaborating the concept, we discuss herein the solution of the simplest forms of ODE, that is of a bar element. By leaving the derivation for later, the ODE of a bar element (as shown in Fig. 1.1) can be given as:

Domain equation

$$EA\frac{d^2u}{dx^2} = -q \tag{1.1}$$

where u and q are the axial displacement of the bar and the external distributed load acting on the bar, respectively. E and A are the Young's modulus and the cross-sectional area of the bar respectively which both are constant for a linear problem. L is the length of the bar and P is an external point load acting at the end of the bar as shown in the Fig. 1.1.

Complementing the domain ODE are the boundary conditions given as follows (which detailed derivation and discussion are delayed until Chapter 2)

Boundary conditions (equations)

$$EA \left.\frac{du}{dx}\right|_{x=L} = P \tag{1.2}$$

$$u\big|_{x=0} = 0 \tag{1.3}$$

The ODE is considered solved when a solution, $u = f(x)$ is found which satisfies all the equations above (i.e. Eqs. (1.2) and (1.3)).

In fact, the exact (closed-formed) solution of the problem can already be obtained by direct integration, thus:

$$u = \left(-\frac{q}{2EA}\right) x^2 + \left(\frac{P + ql}{EA}\right) x \tag{1.4}$$

However, despite the availability of Eq. (1.4), the ODEs of the problem (Eqs. (1.1) to (1.3)) are still discretised numerically herein so as to demonstrate the basic concept of numerical techniques. The problem is chosen due to its simplicity allowing for easy tracing of the discussion.

To convert the ODEs (i.e. Eqs. (1.1) to (1.3)) to its 'equivalent' simultaneous algebraic equations, we start by assuming a guessed solution in the forms of polynomials. In our case, we guess:

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 \tag{1.5}$$

Then, we satisfy Eq. (1.3) by inserting Eq. (1.5) into the equation to give:

$$u|_{x=0} = a_1 + a_2(0) + a_3(0)^2 + a_4(0)^3 + a_5(0)^4 = 0 \tag{1.6}$$

which gives:

$$a_1 = 0 \tag{1.7}$$

Next we satisfy Eq. (1.2) by inserting Eq. (1.5) into the equation to obtain:

$$EA\left.\frac{du}{dx}\right|_{x=L} = EA\left(a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3\right) = P \tag{1.8}$$

Finally, by inserting Eq. (1.5) into Eq. (1.1), the following is obtained:

$$EA\frac{d^2u}{dx^2} + q = EA\left(2a_3 + 6a_4x + 12a_5x^2\right) + q \neq 0 \tag{1.9}$$

Observing Eq. (1.9), it must be noted that, whilst each of Eqs. (1.6) to (1.8) is an act of forcing the equation to a certain values (i.e 0 and P, respectively), hence the "satisfaction" of the equations, the insertion of the guess function (Eq. (1.5)) into the domain equation (Eq. (1.1)) is yet a satisfaction of the original equation hence the use of the inequality symbol ($\neq$). As we are going to see, the forcing of Eq. (1.8) at several locations within the domain to a null value is what satisfies the equation and what creates sufficient number of equations.

By grouping Eqs. (1.7) to (1.9) together, we can see that, so far, we have established three simultaneous equations as follows:

$$a_1 = 0 \tag{1.10a}$$

$$EA\left(a_2 + 2a_3L + 3a_4L^2 + 4a_5L^3\right) = P \tag{1.10b}$$

$$EA\left(2a_3 + 6a_4x + 12a_5x^2\right) + q \neq 0 \tag{1.10c}$$

However, observing Eq. (1.10) we can notice that:

1. We have five (5) unknown constants, a_1, a_2, a_3, a_4 and a_5 but with only three (3) simultaneous equations.

2. The last equation that is Eq. (1.10c) (previously Eq. (1.9)) is still not algebraic but continuous in x. Also, the left hand side of the equation is not equal to ($\neq$) the right hand side because the guessed function is yet the solution of the ODE, as mentioned previously.

So to get the sufficient number of equations (and to convert Eq. (1.10c) into algebraic) we argue that, since Eq. (1.10c) is obtained from domain equation, the equation must hold (must be true) throughout the domain thus we can evaluate Eq. (1.10c) everywhere in the domain as much as we need. In our case, to complement Eqs. (1.10a) and (1.10b), we evaluate Eq. (1.10c) at three (3) locations in the bar, says at $x = L/3, L/2, 2L/3$ to obtain:

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.11a}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.11b}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.11c}$$

Eq. (1.11) are the results of forcing Eq. (1.10c) to a null value at several locations within the domain as mentioned previously. Also, as can be seen, such an act does not only represent the satisfaction of the original ODE, but also convert the ODE into a set of algebraic equations.

Now, by re-grouping Eqs. (1.10a), (1.10b) and (1.11), we then have a sufficient number of algebraic equations, given as:

$$a_1 = 0 \tag{1.12a}$$

$$EAa_2 + 2EAa_3L + 3EAa_4L^2 + 4EAa_5L^3 = P \tag{1.12b}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{3}\right) + 12EAa_5\left(\frac{L}{3}\right)^2 + q = 0 \tag{1.12c}$$

$$2EAa_3 + 6EAa_4\left(\frac{L}{2}\right) + 12EAa_5\left(\frac{L}{2}\right)^2 + q = 0 \tag{1.12d}$$

$$2EAa_3 + 6EAa_4\left(\frac{2L}{3}\right) + 12EAa_5\left(\frac{2L}{3}\right)^2 + q = 0 \tag{1.12e}$$

Eq. (1.12) is thus the 'equivalent' simultaneous algebraic equations of the ODE of the problem which are given originally in Eqs. (1.1) to (1.3). In other words, we can say that:

> *"Eq.* (1.12) *are the 'equivalent' algebraic forms of Eqs.* (1.1) *to* (1.3)*".*

So this is basically the main concept shared by all numerical techniques such as FEM, FDM, BEM and Meshfree. But it is also the character of a numerical technique to treat the equations in matrix forms as this what suits computer programming. In this context, Eq. (1.12) can be arranged in matrix forms as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{3}\right) & 12EA\left(\frac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{L}{2}\right) & 12EA\left(\frac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\frac{2L}{3}\right) & 12EA\left(\frac{2L}{3}\right)^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p \\ -q \\ -q \\ -q \end{Bmatrix} \tag{1.13}$$

or

$$[k]\{u\} = \{r\} \tag{1.14}$$

where

$$[k] = k_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & EA & 2EAL & 3EAL^2 & 4EAL^3 \\ 0 & 0 & 2EA & 6EA\left(\dfrac{L}{3}\right) & 12EA\left(\dfrac{L}{3}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\dfrac{L}{2}\right) & 12EA\left(\dfrac{L}{2}\right)^2 \\ 0 & 0 & 2EA & 6EA\left(\dfrac{2L}{3}\right) & 12EA\left(\dfrac{2L}{3}\right)^2 \end{bmatrix} \tag{1.15a}$$

$$\{u\} = u_j = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{1.15b}$$

$$\{r\} = r_j = \begin{Bmatrix} 0 \\ P \\ -q \\ -q \\ -q \end{Bmatrix} \tag{1.15c}$$

Note: k_{ij}, u_j, and r_j are the alternative notations known as indicial/tensorial notations. We are going to use this notation system and the matrix forms interchangeably.

By solving Eq. (1.12) or Eq. (1.13), we would then obtain the numerical values of a_1, a_2, a_3, a_4 and a_5, and by inserting these values into the original guessed function, Eq. (1.5) we thus obtain the numerical solution of the problem.

2 Galerkin Formulation: Bar Element

2.1 Introduction

In the previous chapter, we have discussed the basic concept of numerical technique and the basic concept of WRM. In this chapter, we are going to discuss on the specific form of WRM that is employed in the present day of FEM that is Galerkin WRM. However, since it has also been mentioned earlier that FEM is nothing but a numerical solution to a PDE, it is important in any FEM endeavour for the analyst to be familiar with the relevant PDE including its derivation and its closed-form solution (if available). This way, when the analyst is required to embark on a new project or study, he or she is already being trained to look at the problem from the first principle, and identify all the relevant aspects or concerns, before he or she employs FEM in getting the solution of the problem.

2.2 Ordinary Differential Equation of Bar Problem

As mentioned, it is vital for an analyst to get into the problem from the first principle and in many cases; this would mean from the derivation of the relevant PDE (or ODE). Since in the previous chapter, we have been introduced to the ODE of bar problem, herein we are going to show the derivation of the ODE. Fig. 2.1(a) shows a bar element subjected to an external distributed load, $q(x)$ and an end load, P and its differential element.

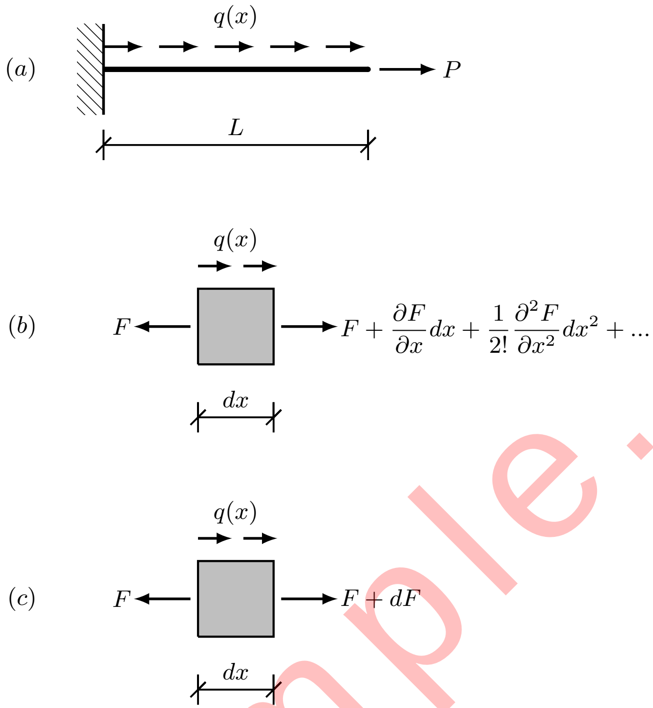

Figure 2.1: Bar structure and its differential element.

It can be argued that, although we have an axial force, F at the left side of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of the axial force at the right side of the differential element must change. However we don't know the exact change of this force, else we would not have this problem in the first place, would we? But by assuming the change is continuous, we can say that the force at the right-side of the differential element can be represented by Taylor series, as shown in Fig. 2.1(b).

But by assuming higher order terms as insignificant and since this is a 1D problem (i.e. $\partial(\quad) = d(\quad)$), a state as shown in Fig. 2.1(c) is considered. This is an important argument thus must be grasped by the readers because as we going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces acting on it, we are in the position to derive the ODE for the bar problem. Since the bar only deform in axial direction, only equilibrium in x-direction need to be considered thus:

$$\sum F_x = 0 = -F + (F + dF) + q\,dx \tag{2.1}$$

By rearranging gives:

$$\frac{dF}{dx} = -q \tag{2.2}$$

From Hooke's Law, we know that:

$$\sigma = E\epsilon \tag{2.3}$$

where σ is the axial stress, E is the Modulus Young and ϵ is the axial strain. Since

From Hooke's Law, we know that:

$$\sigma = \frac{F}{A} \tag{2.4}$$

and

$$\epsilon = \frac{du}{dx} \tag{2.5}$$

By inserting Eq. (2.4) and Eq. (2.5) into Eq. (2.3) gives:

$$F = EA\frac{du}{dx} \tag{2.6}$$

By differentiating Eq. (2.6) once gives:

$$\frac{dF}{dx} = EA\frac{d^2u}{dx^2} \tag{2.7}$$

By inserting Eq. (2.7) into Eq. (2.2) would then give the ODE which we have previously encountered in Chapter 1 (Eq. (1.1)), that is:

$$EA\frac{d^2u}{dx^2} = -q \tag{2.8}$$

In obtaining a unique solution, for every ODE (or PDE for that matter), the domain equation/s must be supplemented by boundary equations, and for this particular case, the equations are given as below:

Natural/force boundary conditions

$$EA\,\frac{du(x)}{dx}\bigg|_{x=0} = F_0 \tag{2.9a}$$

$$EA\,\frac{du(x)}{dx}\bigg|_{x=L} = -F_L \tag{2.9b}$$

Essential/displacement boundary conditions

$$u\big|_{x=0} = u_0 \tag{2.9c}$$

$$u\big|_{x=L} = u_L \tag{2.9d}$$

Note that the sign convention for the natural (force) boundary conditions above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces. Since Eq. (2.8) is a 2nd order ODE, two boundary conditions out of the four given above must be known in prior so as to have a well-posed problem.

Having established the bar ODE, we are now all set to discuss the basic concept of Galerkin WRM. This is given next.

2.3 Fundamental of Galerkin WRM

We have discussed the basic concept of WRM that is, the integration of a function consists of a product of the residual function and a weight function and forcing this integrated value to zero in getting an algebraic function.

The Galerkin WRM differs in the following aspects:

1. The guessed solution is expressed in terms of shape functions, N_i and degree of freedoms, u_i instead of interpolation functions and its

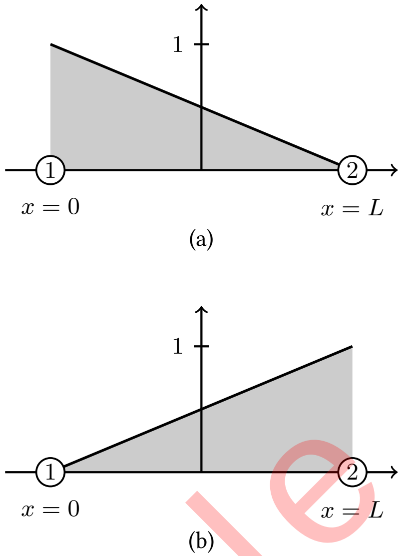

Figure 2.3: Linear shape functions (a) N_1 (b) N_2. The distance, x is measured from node 1.

Both N_1 and N_2 take the shapes as shown in Fig. 2.3. Take note of the shapes as they will be referred later in the discussion of integration by parts and the resulting boundary terms. For example, note that in both cases the shape functions, N_i has a value of unity at node i and zero at other nodes.

2.4.2 Derivation of Quadratic Bar Shape Function

Evaluate Eq. (2.10b) at the location of the nodes (i.e. at both ends, $x = 0$, $x = L/2$ and $x = L$) and equate them according to the dof give:

$$
\begin{aligned}
u(x)|_{x=0} &= a_1 + a_2(0) + a_3(0)^2 = u_1 \\
u(x)|_{x=\frac{L}{2}} &= a_1 + a_2(\frac{L}{2}) + a_3(\frac{L}{2})^2 = u_2 \\
u(x)|_{x=L} &= a_1 + a_2(L) + a_3(L)^2 = u_3
\end{aligned}
\tag{2.17}
$$

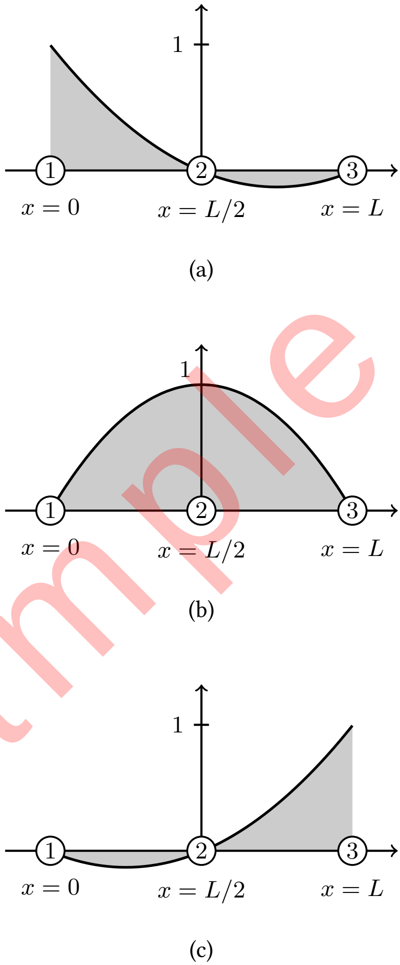

Figure 2.4: Quadratic shape functions (a) N_1 (b) N_2 (c) N_3. The distance, x is measured from node 1.

2.9 Worked Example 2.1

Now, we are going to solve the previous problem of Worked Example 1.1 by considering FEM as illustrated in Fig. 2.7. The problem is solved by the assembly of two bar elements. Both linear and quadratic bars are considered. The results are then verified against the closed-form solution of Eq. (1.4).

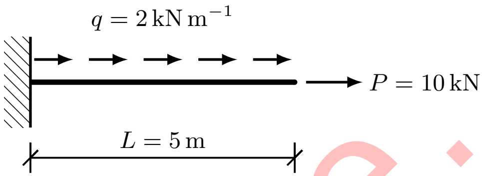

Figure 2.7: Cantilever bar with uniformly distributed load, q and a single point load, P. ($E = 200\times10^6\,\mathrm{kN\,m^{-2}}$ and $A = 0.04\,\mathrm{m^2}$).

2.9.1 Linear Bar Element

Due to the symmetricality, element 1 and element 2 would have a similar local stiffness matrix and load vector, thus:

$$k_{ij}^1 = k_{ij}^2 = \begin{bmatrix} 3.2 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \tag{2.55}$$

and

$$r_i^1 = \begin{Bmatrix} 2.5 + b_1^1 \\ 2.5 + 0 \end{Bmatrix} \tag{2.56}$$

$$r_i^2 = \begin{Bmatrix} 2.5 + 0 \\ 2.5 + 10 \end{Bmatrix} \tag{2.57}$$

Note that b_1^1 is the local reaction at the support of element 1.

The assembled global stiffness matrix, $[K]$ is given as:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$$= \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \tag{2.58}$$

$$= \begin{bmatrix} 3.2 & -3.2 & 0 \\ -3.2 & 6.4 & -3.2 \\ 0 & -3.2 & 3.2 \end{bmatrix} \times 10^6$$

and the assembled load vector, $\{R\}$ is given as:

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} r_1^1 \\ r_2^1 + r_1^2 \\ r_2^2 \end{Bmatrix} = \begin{Bmatrix} 2.5 + b_1^1 \\ 2.5 + 2.5 \\ 2.5 + 10 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{2.59}$$

To emphasize the global reactions forces, b_1^1 is expressed as B_1. This is unknown variable because its corresponding dof is the essential boundary condition which in turn, is a known value. The whole equilibrium equation can thus be given as:

$$\begin{bmatrix} 3.2 & -3.2 & 0 \\ -3.2 & 6.4 & -3.2 \\ 0 & -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2.5 + B_1 \\ 5.0 \\ 12.5 \end{Bmatrix} \tag{2.60}$$

By imposing the essential boundary conditions ($U_1 = 0$), the assembled global equilibrium equation is reduced to:

$$\begin{bmatrix} 6.4 & -3.2 \\ -3.2 & 3.2 \end{bmatrix} \times 10^6 \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 5.0 - 0 \\ 12.5 - 0 \end{Bmatrix} \tag{2.61}$$

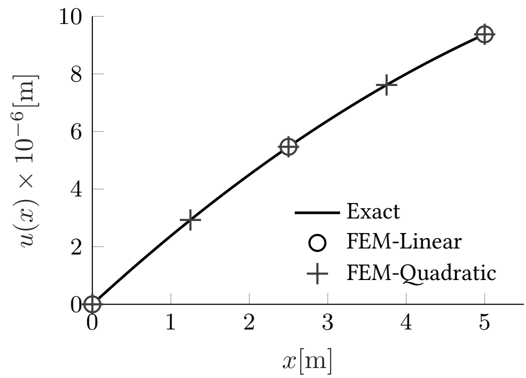

Figure 2.8: Comparison between u_{exact}, u_{linear} and $u_{\text{quadratic}}$

$$\begin{aligned}\{\sigma^2\} &= \frac{\{b^2\}}{A} \\ &= \frac{1}{0.04}\begin{Bmatrix} -15 \\ 0 \\ 10 \end{Bmatrix} \\ &= \begin{Bmatrix} -375 \\ 0 \\ 250 \end{Bmatrix}\end{aligned} \tag{2.86}$$

The validity of both results thus formulation can be assessed by comparing their values with those previously obtained in Worked Example 1.2. The plot of FEM results against the closed-form solution (Eq. (1.4)) are shown in Fig. 2.8 which numerical values at several locations along the bar are given in Table 2.3.

Based on the plot and the table, for the first time the convergence nature of a numerical analysis becomes obvious. As can be seen, the assembled two linear elements provide quite a poor approximation except at the location of the nodes where the results agree with the closed-form solution. A better hence a converged solution is provided by the quadratic elements thus an immediate demonstration of the beneficial effect of the use of higher or-

Table 2.3: Comparison of values between u_{exact}, u_{linear} and $u_{\text{quadratic}}$

x	0 m	1.25 m	2.5 m	3.75 m	5 m
u_{linear}	-		5.47×10^{-6}	-	9.38×10^{-6}
$u_{\text{quadratic}}$	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}
u_{exact}	0	2.93×10^{-6}	5.47×10^{-6}	7.62×10^{-6}	9.38×10^{-6}

der elements. This is an immediate demonstration to the statement *"FEM converges to the 'accurate' solution with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on)"* which was mentioned in the previous chapter.

2.10 Matlab Source Codes

2.10.1 Linear Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;   % Young's modulus (kN/m)
A = 0.04;    % Area (m^2)
L = 5;       % Bar length [m]
P = 10;      % Point load [kN]
q = 2;       % Distributed load [kN/m]

% ---------------------------------------------
% FEM solution - Displacement
% ---------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = A*E/L1*[ 1 -1;
             -1  1];
k2 = A*E/L2*[ 1 -1;
             -1  1];

r1 = [q*L1/2; q*L1/2];
r2 = [q*L1/2; q*L1/2];

% Assemble global matrix, K and vector, R
K = zeros(3);
```

```
K(1:2, 1:2) = K(1:2, 1:2) + k1;
K(2:3, 2:3) = K(2:3, 2:3) + k2;

R = zeros(3,1);
R(1:2) = R(1:2) + r1;
R(2:3) = R(2:3) + r2;

% Point load, P at the bar end
R(3) = R(3)+P;

% Solve for global displacement
U      = zeros(3,1);
U(2:3) = K(2:3,2:3)\R(2:3);

% -----------------------------------------
% FEM solution - Reaction, b & stress, sig
% -----------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2)] - r1;
b2 = k2*[U(2); U(3)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```

2.10.2 Quadratic Bar Element

```
% Clear data
clc; clear; close all

% Input
E = 200e6;  % Young's modulus (kN/m)
A = 0.04;   % Area (m^2)
L = 5;      % Bar length [m]
P = 10;     % Point load [kN]
q = 2;      % Distributed load [kN/m]

% -----------------------------------------
% FEM solution - Displacement
% -----------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = A*E*[ 7/(3*L1) -8/(3*L1)  1/(3*L1);
          -8/(3*L1) 16/(3*L1) -8/(3*L1);
           1/(3*L1) -8/(3*L1)  7/(3*L1)];
k2 = A*E*[ 7/(3*L2) -8/(3*L2)  1/(3*L2);
          -8/(3*L2) 16/(3*L2) -8/(3*L2);
           1/(3*L2) -8/(3*L2)  7/(3*L2)];

r1 = [q*L1/6; 2*q*L1/3; q*L1/6];
r2 = [q*L2/6; 2*q*L2/3; q*L2/6];
```

```
% Assemble global matrix, K and vector, R
K = zeros(5);
K(1:3, 1:3) = K(1:3, 1:3) + k1;
K(3:5, 3:5) = K(3:5, 3:5) + k2;

R = zeros(5,1);
R(1:3) = R(1:3) + r1;
R(3:5) = R(3:5) + r2;

% Point load, P at the bar end
R(5) = R(5)+P;

% Solve for global displacement
U      = zeros(5,1);
U(2:5) = K(2:5,2:5)\R(2:5);

% ------------------------------------------
% FEM solution - Reaction, b & stress, sig
% ------------------------------------------

% Internal force or reaction
b1 = k1*[U(1); U(2); U(3)] - r1;
b2 = k2*[U(3); U(4); U(5)] - r2;

% Internal stress
sig1 = b1/A;
sig2 = b2/A;
```


2.11 Excercises

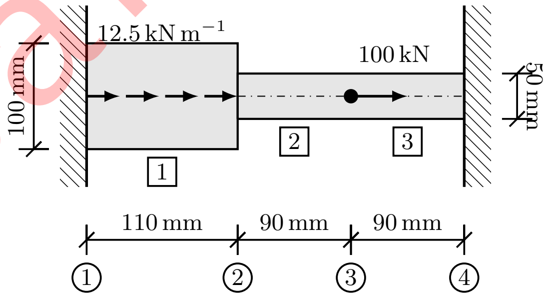

Figure 2.9

1. The rod structure of circular cross section shown in Fig. 2.9 is subjected to a concentrated force of 100 kN and a traction force of 12.5 kN m^{-1}. The material for all rods is aluminium with E = 70 GPa. By using three 2-node bar elements, determine;

3 Galerkin Formulation: Beam Element

3.1 Introduction

In the previous chapter, we have discussed the concept of Galerkin WRM hence FEM and formulated the discretised equation for bar element. Herein, we are going to continue our discussion by formulating the discretised equation for beam problem. We begin by deriving the Euler-Bernoulli differential equation of beam which is given next.

3.2 Ordinary Differential Equation of Euler-Bernoulli Beam

An Euler-Bernoulli beam is a structural member that resists loads by bending and shearing. The corresponding deformation would be rotation and translation. Consider a structural beam which is subjected to a distributed load, $q(x)$ as shown in Fig. 3.1(a). The differential equation for such a beam can be derived for static loading by considering its differential element as shown in Fig. 3.1(b). Also shown is the typical arrangement of a beam structure.

As argued previously for bar element (in Chapter 2), although we have moment force, M and shear force, V at the left side of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of these forces at the right side of the differential element must change. However we don't know the exact change of these forces, else we would not have the problem in the first place would we? But by assuming the change is continuous, we can say that the forces at the right-side of the differential element can be represented by a Taylor series, as shown in Fig. 3.1(b).

By assuming higher order terms as insignificant and since this is a 1D problem (i.e.$\partial(\quad) = d(\quad)$), a state as shown in Fig. 3.1(c) is considered. This is an important argument thus must be grasped by the readers because as we are going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces acting on it, we are in the position to derive the ODE for beam problem. However, it must be noted again that the following differential equation does not consider axial deformation thus the absent of axial forces. Beams allowing such forces are called beam-column , of which the FEM formulation is given in the next chapter. Also, present formulation assumes slope is equalled to rotation. A more general formulation would be the Timoshenko Beam Theory as it allows different values for the two entities, but it is not included in our discussion.

Based on Fig. 3.1(c), the following equilibrium of forces can be employed:

$$\sum F_x = 0 \tag{3.1}$$

$$\sum F_y = 0 \tag{3.2}$$

$$\sum M_z = 0 \tag{3.3}$$

which yield

$$V - (V + dV) - q\,dx = 0 \tag{3.4}$$

$$-M + (M + dM) - V\,dx - q\frac{dx^2}{2} = 0 \tag{3.5}$$

where q is the distributed transverse external loading and w the deflection of the beam.

3.7 Imposition of Essential Boundary Conditions

Essential (or displacement) boundary conditions must be imposed before the equilibrium equation of the beam can be solved. Basic concept of direct imposition of boundary conditions has been described in detail in previous chapter.

Eq. (3.12) give all the possible essential boundary conditions of a beam, reproduced herein for ease of reading as:

$$\left.\frac{dw}{dx}(x)\right|_{x=0} = \theta_0 \tag{3.50a}$$

$$\left.\frac{dw}{dx}(x)\right|_{x=L} = \theta_L \tag{3.50b}$$

$$w(x)|_{x=0} = w_0 \tag{3.50c}$$

$$w(x)|_{x=L} = w_L \tag{3.50d}$$

Graphical representations of these boundary conditions are given in Fig. 3.6.

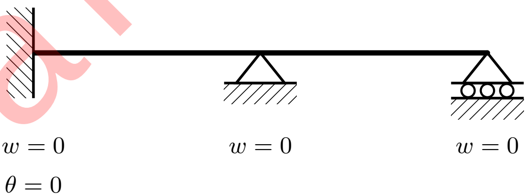

Figure 3.6: Various beam's essential boundary conditions.

3.8 Worked Example 3.1

Let's put what we have learned so far into practice. Fig. 3.7 shows a cantilever beam subjected to a distributed load of $2\,\mathrm{kN\,m^{-1}}$ and an edge point load of $-5\,\mathrm{kN}$.

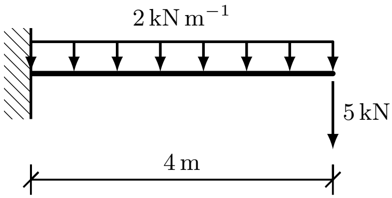

Figure 3.7: Beam with distributed and point loads. ($E = 200\times10^6$ kN m^{-1} and $I = 1.333\times10^{-4}$ m^4).

An analytical solution for the beam can be obtained by conducting integration directly on the ODE given by Eq. (3.10) and satisfying the all the relevant boundary conditions (both force and displacement in Eqs. (3.11) and (3.12), which are:

$$EI\left.\frac{d^3w}{dx^3}\right|_{x=4} = -V_L = -(-5\,\mathrm{kN}) \tag{3.51a}$$

$$EI\left.\frac{d^2w}{dx^2}\right|_{x=4} = -M_L = 0 \tag{3.51b}$$

$$\left.\frac{dw}{dx}(x)\right|_{x=0} = \theta_0 = 0 \tag{3.51c}$$

$$w(x)|_{x=0} = w_0 = 0 \tag{3.51d}$$

The analytical or the closed form solution of the beam can be given as:

$$w_{\text{exact}} = \frac{qx^4}{24EI} + \frac{(-P-qL)x^3}{6EI} - \frac{qL^2x^2}{4EI} - \frac{(-P-qL)Lx^2}{2EI} \tag{3.52}$$

Herein, we are going to solve the problem by the assembly of two beam elements of equal length and compare the results against Eq. (3.52) to validate the FEM formulation. Due to the symmetrically, element 1 and element 2 would have a similar local stiffness matrix thus:

$$k^1_{ij} = k^2_{ij} = \begin{bmatrix} 4.00 & 4.00 & -4.00 & 4.00 \\ 4.00 & 5.33 & -4.00 & 2.67 \\ -4.00 & -4.00 & 4.00 & -4.00 \\ 4.00 & 2.67 & -4.00 & 5.33 \end{bmatrix} \times 10^4 \tag{3.53}$$

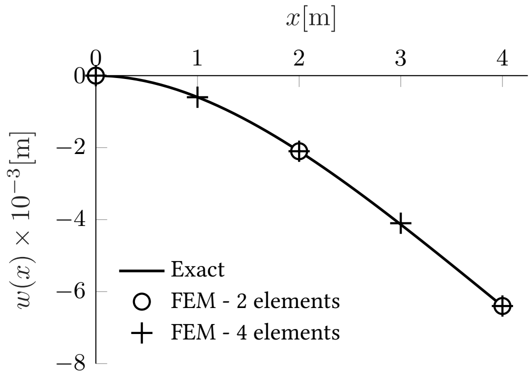

Figure 3.9: Plot of w_{FEM} and w_{exact}.

Table 3.4: Comparison of values of reaction forces (in kN) at fix support.

	B_1	B_2
$w_{\text{FEM-2 elements}}$	13	36
$w_{\text{FEM-4 elements}}$	13	36
Software (converged)	13	36

with the increase in the order of polynomial of the guessed functions (or degree of freedom and mesh density later on)" which was stated in previous chapter.

3.9 Matlab Source Codes

```
% Clear data
clc; clear; close all

%Input
E = 200e6;       % Young's modulus [kN/m]
I = 1.333e-4;    % Moment of inertia [m^4]
L = 4;           % Bar length [m]
P = -5;          % Point load [kN]
q = -2;          % Distributed load [kN/m]
```

```
% -------------------------------------------
% FEM solution - Displacement
% -------------------------------------------

% Elements length
L1 = L/2; L2 = L/2;

% Matrix k & force r
k1 = [12*E*I/L1^3  6*E*I/L1^2 -12*E*I/L1^3  6*E*I/L1^2;
       6*E*I/L1^2  4*E*I/L1    -6*E*I/L1^2  2*E*I/L1;
     -12*E*I/L1^3 -6*E*I/L1^2  12*E*I/L1^3 -6*E*I/L1^2;
       6*E*I/L1^2  2*E*I/L1    -6*E*I/L1^2  4*E*I/L1];
k2 = [12*E*I/L1^3  6*E*I/L1^2 -12*E*I/L1^3  6*E*I/L1^2;
       6*E*I/L1^2  4*E*I/L1    -6*E*I/L1^2  2*E*I/L1;
     -12*E*I/L1^3 -6*E*I/L1^2  12*E*I/L1^3 -6*E*I/L1^2;
       6*E*I/L1^2  2*E*I/L1    -6*E*I/L1^2  4*E*I/L1];

r1 = [q*L1/2; q*L1^2/12; q*L1/2; -q*L1^2/12];
r2 = [q*L2/2; q*L2^2/12; q*L2/2; -q*L2^2/12];

% Assemble global matrix, K and vector, R
K = zeros(6);
K(1:4,1:4) = K(1:4,1:4) + k1;
K(3:6,3:6) = K(3:6,3:6) + k2;

R = zeros(6,1);
R(1:4) = R(1:4) + r1;
R(3:6) = R(3:6) + r1;

% Point load, P
R(5) = R(5) + P;

% Solve for global displacement
D      = zeros(6,1);
D(3:6) = K(3:6,3:6)\R(3:6);

% -------------------------------------------
% FEM solution - Internal reaction, b
% -------------------------------------------

% Internal force or reaction
b1 = k1*[D(1); D(2); D(3); D(4)] - r1;
b2 = k2*[D(3); D(4); D(5); D(6)] - r2;
```


3.10 Exercises

1. The displacement function for beam element as shown in Fig. 3.10(a) is assumed as:

$$v = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

   i. Explain the steps that need to be taken to determine the values

4 Plane Structures: Truss and Frame

4.1 Introduction

In the previous chapters, we have discussed the derivation of FEM formulation for bar and beam elements. However, these elements were arranged and assembled in a line. A more general arrangement would require the elements to be arbitrarily oriented and assembled. Assembly of such oriented elements would make up a truss system and a frame system, respectively. As we are going to see, the orientation process requires the introduction of the transformation matrix. The use of this matrix is to transform local entities into global entities. Another point to emphasize is the introduction of two degree of freedoms and two load components into the bar element's global representation. Also, since the construction of a frame would require the transfer of axial load/force, a beam element is supplied with extra degree of freedoms in the axial direction. As can be seen, this will involve the combination of previously derived bar and beam elements to form what is called beam-column element.

4.2 Truss System

A truss system is an assembly of inclined bar elements. Figure 4.1 shows a typical arrangement of a plane truss system.

To allow for the inclination of the truss members and the corresponding assembly, a bar element formulation must be re-expressed in a global manner. In a global axis, a bar element would have two degree of freedoms per node. Such a transformation requires us to establish what is known as a

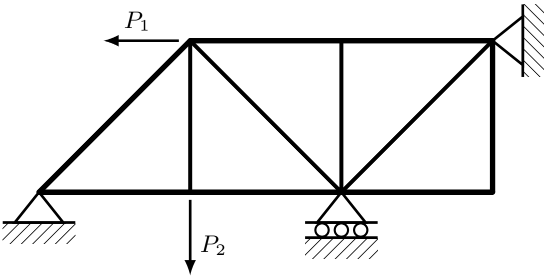

Figure 4.1: A typical truss system.

transformation matrix as discussed next.

4.2.1 Bar Transformation Matrix

Consider an inclined bar as shown in Fig. 4.2, together with the newly introduced elemental global direction dofs and previously defined local dofs. Note that, to distinguish between local and elemental global direction dofs, the former is primed.

For a linear bar (two-nodes bar), by considering the geometry of Fig. 4.2 the relationship between the local and the elemental global dofs can be given as:

$$u'_1 = u_1 \cos\beta + u_2 \sin\beta \tag{4.1a}$$

$$u'_2 = u_3 \cos\beta + u_4 \sin\beta \tag{4.1b}$$

In matrix forms, Eq. (4.1a) can be given as:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{4.2}$$

or

$$\{u'\} = [T]\{u\} \tag{4.3}$$

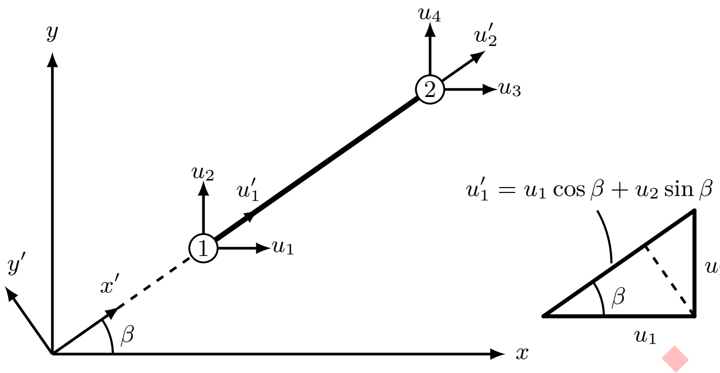

Figure 4.2: Degree of freedoms of inclined bar element

where $\{u'\}$ is the vector of local dofs, $[T]$ is from now on is called the bar transformation matrix and $\{u\}$ is the vector of the elemental global dofs of the bar element.

To simplify the expression, $[T]$s is expressed as:

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \tag{4.4}$$

where

$$c = \cos\beta$$

$$s = \sin\beta$$

So far, we have established the relationship between local dofs of bar and its elemental global dofs. Now we are going to establish the relationship between local forces and the elemental global forces. By referring to Fig. 4.3, the relationship can be given as:

$$r_1 = r'_1 \cos\beta \tag{4.5a}$$

$$r_2 = r'_1 \sin\beta \tag{4.5b}$$

$$r_3 = r'_2 \cos\beta \tag{4.5c}$$

$$r_4 = r'_2 \sin\beta \tag{4.5d}$$

5 Eigenproblems: Free Vibration and Buckling

5.1 Introduction

In previous chapters, we have been dealing with structures which are loaded in the direction of the degree of freedoms; point loads (either equivalent or nodal loads) in the direction of the translational dof or applied moment in the direction of the rotational dof. Herein, we are going to discuss a quite different situation that is, the deformation which is 'ungoverned' by loads that are acting on the structure.

5.2 "Ungoverned by the Loads" and Eigenproblems

The word "ungoverned", however, requires further elaboration. The word refers to the statement that the load vector, $\{r\}$ or $\{R\}$ is set to zero. Mathematically this means solving only the homogenous part of the differential equation of the problem. The resulting values would be some properties of the structure and their corresponding deformation.

For example, free vibration refers to the "vibration" of a structure which is described by the natural frequencies and the corresponding mode shapes, without any explicit consideration on the type of external loading. Only during the discussion of resonance, these frequencies would then be compared with the incoming frequencies (external loads).

Same goes to the discussion of buckling of a structure. The load that would cause the buckling will be specific critical values of compressive axial force termed as buckling loads . As will be seen, since Euler-Bernoulli beam formulation able to capture such a phenomenon and determine the buckling loads and their corresponding buckling modes without the need to introduce axial dofs, it should be obvious that the bucking loads and their modes are not governed by the external applied loads.

Since both values (natural frequencies, buckling loads) are not governed by the loads, they must be some properties of the structure hence the name "eigen" which means "inherent" or "characteristic" in German. Then, what governs their values? Something must affect their values, must not they? As some properties, they are governed by other properties of the structures; material and geometrical properties. In our discussion on bar, beam and their inclined elements, these would be Young's modulus, E, cross-sectional area, A, second moment of area, I and element's length, L.

Also, since these values are "ungoverned" by the loads, we will see that the FEM formulation for both problems will involve with the discretization of their differential equations without the forcing terms hence the setting up of load vector, $\{r\}$ or $\{R\}$ to zero.

Accordingly, all physical problems that fall under the same argument are called eigenproblems as their equilibrium equations can all be arranged into a standard mathematical statement. If $[A]$ and $[B]$ are square matrices with known coefficients and λ is an unknown constant, an eigenproblem is a problem that can be described by the following typical mathematical statement:

$$([A] + \lambda[B])\{d\} = 0 \tag{5.1}$$

where λ is termed as eigenvalue and $\{d\}$ is termed as eigenvector.

For free vibration analysis, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the mass matrix, $[M]$, constant λ represents the square of natural frequencies and vector $\{d\}$ represents the vector of mode shape dofs $\{\hat{d}\}$.

For buckling problem, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the stress stiffness matrix $[K_G]$, constant λ represents the buckling load, P and vector $\{d\}$ represents the buckling modes . Table 5.1 summarizes these.

Table 5.1: Eigenproblem grouping

	Free vibration	Buckling
$[A]$	$[K]$	$[K]$
$[B]$	$[M]$	$[K_G]$
λ	λ	P
$\{d\}$	$\{d\}$	$\{d\}$
$([A]+\lambda[B])\{d\}=0$	$([K]-\lambda[M])\{d\}=0$	$([K]+P[K_G])\{d\}=0$

Besides having a typical statement, eigenproblem also has a typical argument for the solution that is;

> "*The solution of Eq.* (5.1) *has two solutions, the trivial solution when* $\{d\} = 0$ *and the non-trivial solution when the determinant of the coefficient matrices is zero i.e.* $[A]+\lambda[B]=0$. *Since a trivial solution would also mean that there will be no deformation, a non-trivial solution is therefore the solution of interest.*"

Since the non-trivial solution is of interest, the setting up of

$$|[A]+\lambda[B]|=0 \tag{5.2}$$

will result in a characteristic polynomial. If the coefficient matrices are 4×4 matrix, the resulting characteristic polynomial will be in the forms:

$$\lambda^4+\Phi_4\lambda^3+\Phi_3\lambda^2+\Phi_2\lambda+\Phi_1=0 \tag{5.3}$$

where Φ_1, Φ_2, Φ_3, and Φ_4 are constants resulted from the process of setting up the determinant to zero. For $n \times n$ matrices, the resulting polynomial would be

$$\lambda^n+\Phi_n\lambda^{n-1}+...+\Phi_1\lambda^0=0 \tag{5.4}$$

The four roots of Eq. (5.3) and the n roots of Eq. (5.4) are thus the eigenvalues for the problem. For free vibration, these would be the natural frequencies and for buckling problem, these would be the buckling loads. Inserting one of the roots (or eigenvalues) back into Eq. (5.1) will allow the eigenvector for that particular eigenvalue to be solved hence determined. For

Table 5.2: Comparison of ω values between FEM and exact solution

	FEM			Exact
	2 elements	4 elements	8 elements	
1^{st} mode	20.59	20.58	20.58	20.58
2^{nd} mode	130.08	129.13	128.99	128.96

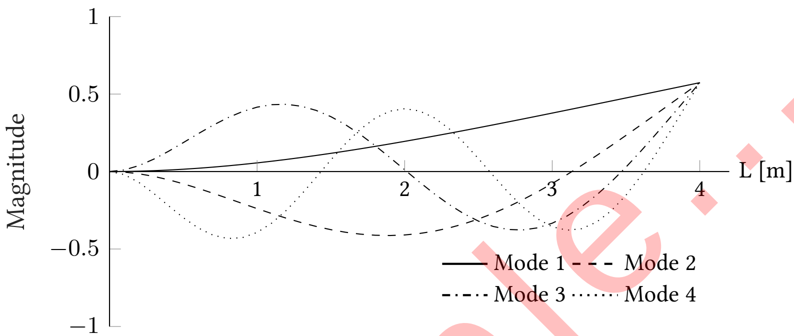

Figure 5.4: Mode shapes of beam's free vibration

5.5 Buckling of Beam

Beam buckling is a phenomenon of instability. The phenomenon is characterized by the out-of-plane deformation; a deformation that is away from the plane in which the buckling load is acting.

The analysis of buckling of structures, on the other hand, refers to the determination of buckling (critical) load, P and its corresponding mode shape, $\{d\}$. If the compressive load within a structure equals this buckling load, buckling will occur. In the context of ultimate limit state design, occurrence of buckling is considered as premature failure as it prevents the attainment of the ultimate resistances of the structure.

The determination of buckling load and its corresponding mode will be done in the same manner as outlined previously for free vibration of beam as the analysis falls under the same category of eigenproblem. By referring to Table 5.1, it can be seen that the process requires the derivation of

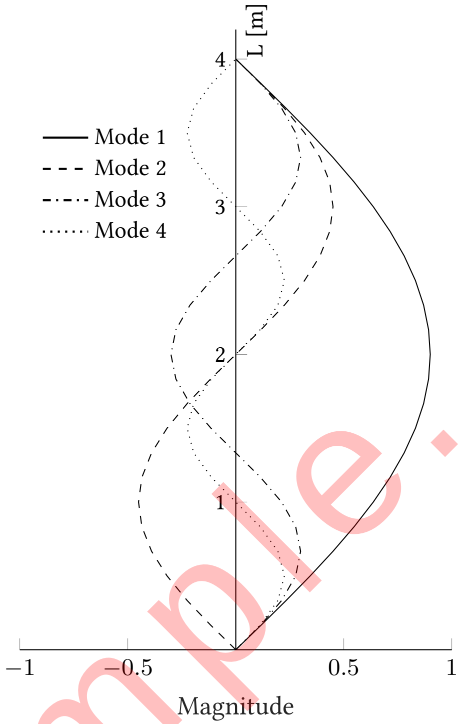

Figure 5.7: The first four buckling modes of a column with pinned at both ends.

is primed so as to distinguish them from their elemental):

$$[m'] = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \tag{5.90}$$

Having established the beam-column's mass matrix, $[m']$ as given by Eq. (5.90), whereas the beam-column's stiffness matrix, $[k']$ is already

6 Dynamic: Forced Vibration

6.1 Introduction

Dynamic analysis is a topic where the main concern is to determine structure's responses or motions when subjected to time-varying loading (or/and time-varying boundary conditions). It is an act of solving the equation of motion of a problem. Accordingly, in this chapter, we are going to focus our discussion on the solution of forced vibration problems for bar and beam elements, as well as frame structures. Although we have derived the discretised equation of motion of beam element in Chapter 5, for completeness, we are going to begin this chapter with the derivation of the discretised equation for bar element, starting from its PDE derivation.

6.2 Bar's PDE of Motion

Derivation of the bar's PDE of motion has been given in Section 5.3.1. We are not going to repeat the procedure but to give directly the PDE as (previously given by Eq. (5.10)):

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = -q(x,t) \tag{6.1}$$

Although we have discretised this PDE in Section 5.3.2, but for completeness, we are going to detail the discretization all over again, this time with the existence of the forcing terms i.e. $q(x,t)$. As a reminder, in previous discretization, this forcing term was omitted because we were considering the homogenous solution only (eigenproblem). Before we proceed, below are the natural and essential boundary conditions of the problem (previously given by Eq. (2.9a)). However, since this is a dynamic analysis, there is a possibility for the boundary conditions to vary with time, thus:

Natural/force boundary conditions

$$EA\frac{du}{dx}\bigg|_{x=0} = F_0(t) \tag{6.2a}$$

$$EA\frac{du}{dx}\bigg|_{x=L} = -F_L(t) \tag{6.2b}$$

Essential/displacement boundary conditions

$$u|_{x=0} = u_0(t) \tag{6.3a}$$

$$u|_{x=L} = u_L(t) \tag{6.3b}$$

Note that the sign convention for the natural (force) boundary conditions above is based on the assumption that the boundary conditions are the reactions at support hence the opposite directions to the internal forces. Since Eq. (6.1) is a second order ODE in x, two boundary conditions out of the four given above must be known in prior so as to have a well-posed problem. Also, since Eq. (6.1) consists of a second order time derivative, two initial conditions must be known. We are going to detail on this later during the discussion of time integration by finite difference method.

6.2.1 Discretization of Bar's PDE of Motion by Galerkin Method

Like previous elements, the discretization process begins with the provision of the interpolation function (for linear bar element):

$$u(x) = N_1u_1 + N_2u_2 \tag{6.4a}$$

or in component forms as:

$$u(x) = N_ju_j \tag{6.4b}$$

where N_j and u_j are the shape functions and degree of freedoms as already discussed in previous chapter for linear bar element (Eq. (2.16)).

By inserting Eq. (6.4a) into Eq. (6.1) gives:

$$EA\frac{\partial^2(N_1u_1+N_2u_2)}{\partial x^2}-\rho A\frac{\partial^2(N_1u_1+N_2u_2)}{\partial t^2}\neq -q(x,t) \tag{6.5a}$$

or in component forms as:

$$EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}\neq -q(x,t) \tag{6.5b}$$

Having established the PDE in terms of shape functions and dofs, we then multiply Eq. (6.5a) with weight functions, N_i consecutively and integrate the inner product so as to obtain the discretised equation.

Thus:

$$\int_0^L N_i\left(EA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\frac{\partial^2(N_ju_j)}{\partial t^2}+q(x,t)\right)dx=0$$

$$\Rightarrow \int_0^L N_iEA\frac{\partial^2(N_ju_j)}{\partial x^2}-\rho A\int_0^L N_i\frac{\partial^2(N_ju_j)}{\partial t^2}+\int_0^L N_iq(x,t)\,dx=0 \tag{6.6}$$

Next, we conduct integration by parts (IBP) to the first term. It must be noted that, no IBP is conducted to the second term because the term is not a spatial x derivative, but a time, t derivative instead. By conducting the IBP, we obtain:

$$\int_0^L \frac{\partial N_i}{\partial x}EA\frac{\partial(N_ju_j)}{\partial x}+\rho A\int_0^L N_iN_j\frac{\partial^2 u_j}{\partial t^2}=\int_0^L N_iq(x,t)\,dx+b_i(t) \tag{6.7}$$

where $b_i(t)$ is the boundary terms (conditions) which can be either Eqs. (6.2) and (6.3) as already explained in detailed in Section 2.5 (Eqs. (2.27) and (2.33)) except that, this time, they can vary with time, t.

While the first integral term of Eq. (6.7) is familiar to us from the discussion of bar element (static problem) in Chapter 2, the second term, as mentioned, deserves further explanation. In this term, the shape functions, N_j is taken out from the derivative since it would not involve with the differentiation because the shape functions are not a function of time, t but instead, a function of x only. But something must be a function of time else the second term would vanish. This leaves us with nothing but the degree of and solve for $\{D\}^4$ gives:

$$\{D\}^4 = \begin{Bmatrix} -4.35\times10^{-8} \\ 1.25\times10^{-8} \\ -1.22\times10^{-8} \\ -9.40\times10^{-8} \\ 2.98\times10^{-7} \\ 4.94\times10^{-8} \\ -1.07\times10^{-6} \\ -9.14\times10^{-7} \\ 2.72\times10^{-8} \\ -3.98\times10^{-7} \\ 3.47\times10^{-6} \\ -2.10\times10^{-7} \end{Bmatrix}$$

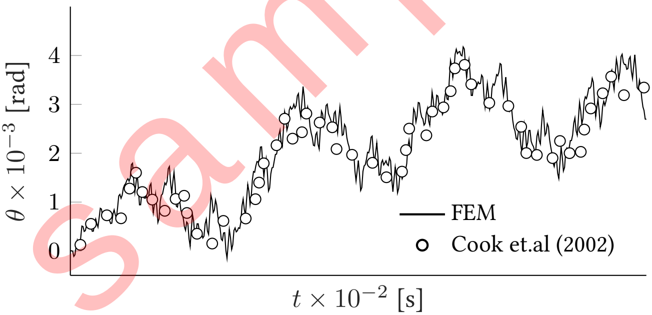

Figure 6.3: Vibration (rotation) of the frame.

These steps can be continued until a specific period of time by repeating the same procedure.

Fig. 6.3 shows the vibration (rotation) of the frame taken at $x = 1\,\mathrm{m}$ from the roller support. The results is compared with Cook et.al (2002)[†].

[†]Cook, R. D., Malkus, D. S., Plesha, M. E., & Witt, R. J. (1974). *Concepts and applications of finite element analysis* (Vol. 4). New York: Wiley. (Figure 11.18-2(d), page 441)

6.7 Matlab Source Code

```
% Clear data
clc; clear; close all

%Input
E   = 200e9;     % Young's modulus [N/m]
A   = 0.01;      % Area [m^2]
I   = 8.33e-6;   % Moment of inertia [m^4]
rho = 7860;      % Density [kg/m^3]
P   = 1e5;       % Point load [N]

dt  = 0.00001;   % Time step [s]
t   = 0:dt:0.1;  % Time vector [s]

% ---------------------------------------------
% FEM solution - Forced vibration
% ---------------------------------------------

% Elements length [m]
L1 = 1.5; L2 = 1.5; L3 = 1.0; L4 = 1.0;

% Elements angle [degree]
th1 = 90; th2 = 90; th3 = 0; th4 = 0;

% Calculate Stifness
k1 = [
 A*E/L1  0            0            -A*E/L1    0            0;
 0        12*E*I/L1^3  6*E*I/L1^2   0         -12*E*I/L1^3  6*E*I/L1^2;
 0        6*E*I/L1^2   4*E*I/L1     0         -6*E*I/L1^2   2*E*I/L1;
-A*E/L1  0            0             A*E/L1    0            0;
 0       -12*E*I/L1^3 -6*E*I/L1^2   0          12*E*I/L1^3 -6*E*I/L1^2;
 0        6*E*I/L1^2   2*E*I/L1     0         -6*E*I/L1^2   4*E*I/L1];
k2 = [
 A*E/L2  0            0            -A*E/L2    0            0;
 0        12*E*I/L2^3  6*E*I/L2^2   0         -12*E*I/L2^3  6*E*I/L2^2;
 0        6*E*I/L2^2   4*E*I/L2     0         -6*E*I/L2^2   2*E*I/L2;
-A*E/L2  0            0             A*E/L2    0            0;
 0       -12*E*I/L2^3 -6*E*I/L2^2   0          12*E*I/L2^3 -6*E*I/L2^2;
 0        6*E*I/L2^2   2*E*I/L2     0         -6*E*I/L2^2   4*E*I/L2];
k3 = [
 A*E/L3  0            0            -A*E/L3    0            0;
 0        12*E*I/L3^3  6*E*I/L3^2   0         -12*E*I/L3^3  6*E*I/L3^2;
 0        6*E*I/L3^2   4*E*I/L3     0         -6*E*I/L3^2   2*E*I/L3;
-A*E/L3  0            0             A*E/L3    0            0;
 0       -12*E*I/L3^3 -6*E*I/L3^2   0          12*E*I/L3^3 -6*E*I/L3^2;
 0        6*E*I/L3^2   2*E*I/L3     0         -6*E*I/L3^2   4*E*I/L3];
k4 = [
 A*E/L4  0            0            -A*E/L4    0            0;
 0        12*E*I/L4^3  6*E*I/L4^2   0         -12*E*I/L4^3  6*E*I/L4^2;
 0        6*E*I/L4^2   4*E*I/L4     0         -6*E*I/L4^2   2*E*I/L4;
-A*E/L4  0            0             A*E/L4    0            0;
 0       -12*E*I/L4^3 -6*E*I/L4^2   0          12*E*I/L4^3 -6*E*I/L4^2;
 0        6*E*I/L4^2   2*E*I/L4     0         -6*E*I/L4^2   4*E*I/L4];


%Calculate Consistent Mass Matrix
m1 = rho*A*L1/420* [140     0       0       70      0        0;
```

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