APPENDIX F

DYNAMIC RESPONSE OF INSTRUMENT SYSTEMS

In our discussions in this text, any temporal variations in the measured quantity have been treated as random variations that contribute to the random uncertainty in the measurement. However, it is also necessary for us to consider additional error sources due to the response of instruments to dynamic, or changing, inputs. An instrument may produce an output with both amplitude and phase (time lag) errors when a dynamic input is encountered.

These dynamic response errors are similar to the variable but deterministic bias errors discussed in Appendix E, and they can be very important in the analysis of a timewise, transient experiment. In the following sections we present the fundamentals needed to estimate these amplitude and phase errors.

F-1 GENERAL INSTRUMENT RESPONSE

The traditional way to investigate the dynamic response of an instrument is to consider the differential equation that describes the output. We assume that the instrument response can be modeled using a linear ordinary differential equation with constant coefficients [1]

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = bx(t) \tag{F.1}$$

where y is the instrument output, x is the input, and n is the order of the instrument.

Instrument response to three different inputs will be discussed: (1) a step change, (2) a ramp input, and (3) a sinusoidal input. These are illustrated in

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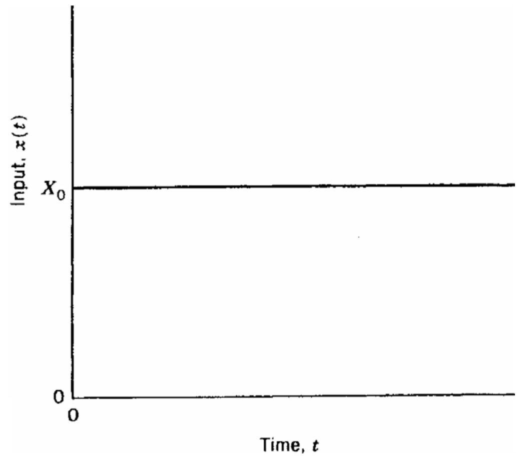

Figure F.1 Step change in input to an instrument.

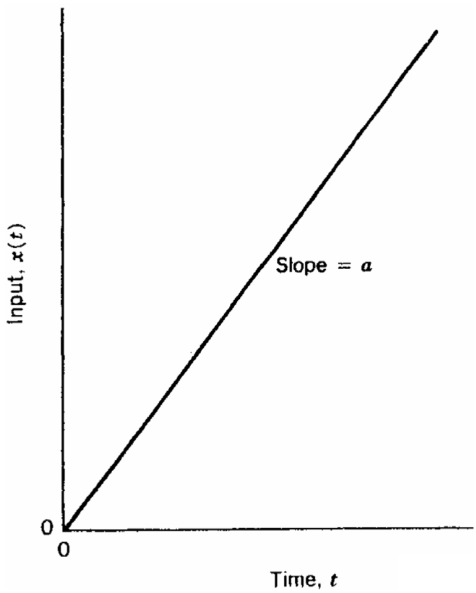

Figure F.2 Ramp change in input to an instrument.

Figures F.1, F.2, and F.3. Mathematically, these inputs are described as follows:

1. Step change $\quad x = 0 \quad t < 0$

$$x = x_0 \quad t > 0 \tag{F.2}$$

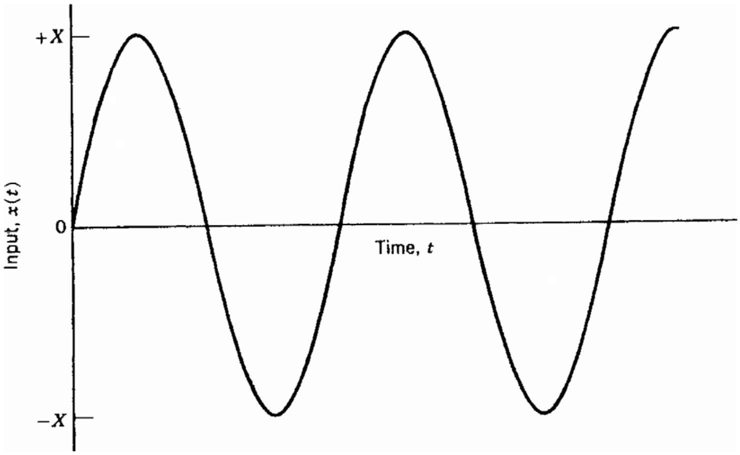

Figure F.3 Sinusoidally varying input to an instrument.

2. Ramp	$x = 0$	$t < 0$	(F.3)
	$x = at$	$t \geq 0$	
3. Sinusoidal	$x = X \sin(\omega t)$	$t > 0$	(F.4)

The response of zero-, first-, and second-order instruments to these inputs are considered next.

F-2 RESPONSE OF ZERO-ORDER INSTRUMENTS

Since $n = 0$ for a zero-order instrument, Eq. (F.1) reduces to an algebraic equation:

$$y = Kx(t) \tag{F.5}$$

where K $(= b/a_0)$ is called the *static gain*. Equation (F.5) shows that the output is always proportional to the input, so there is no error in the output due to the dynamic response. Of course, there will be static errors of the types we have discussed previously.

An example of a zero-order instrument is an electrical resistance strain gauge. The input strain ϵ causes the gauge resistance to change by an incremental amount ΔR according to the relationship [2]

$$\Delta R = FR\epsilon \tag{F.6}$$

where F is the gauge factor and R is the resistance of the gauge wire in the unstrained condition. Since the instrument itself, the gauge wire, is experiencing the input strain directly, there is no dynamic response error in the output.

F-3 RESPONSE OF FIRST-ORDER INSTRUMENTS

The response equation for first-order instruments is usually written in the form

$$\tau \frac{dy}{dt} + y = Kx \tag{F.7}$$

where τ $(= a_1/a_0)$ is the time constant and K $(= b/a_0)$ is the static gain. The definition of a first-order instrument is one that has a dynamic response behavior that can be expressed in the form of Eq. (F.7) [3].

A first-order instrument experiences a time delay between its output and a time-varying input. An example is a thermometer or thermocouple that must undergo a heat transfer process for its reading to respond to a changing input temperature.

The response of a first-order instrument to a step change is found by solving Eq. (F.7) using Eq. (F.2) for x and the initial condition that $y = 0$ at $t = 0$. This solution can be expressed in the form

$$\frac{y}{Kx_0} = 1 - e^{-t/\tau} \tag{F.8}$$

which is plotted in Figure F.4. In one time constant, the response achieves 63.2% of its final value. One must wait for four time constants (4τ) before the response y will be within 2% of the final value.

The response to a ramp input is found by solving Eq. (F.7) using Eq. (F.3) for x and the initial condition that $y = 0$ at $t = 0$. This solution is

$$y = Ka[t - \tau(1 - e^{-t/\tau})] \tag{F.9}$$

which can also be expressed as

$$y - Kat = -Ka\tau(1 - e^{-t/\tau}) \tag{F.10}$$

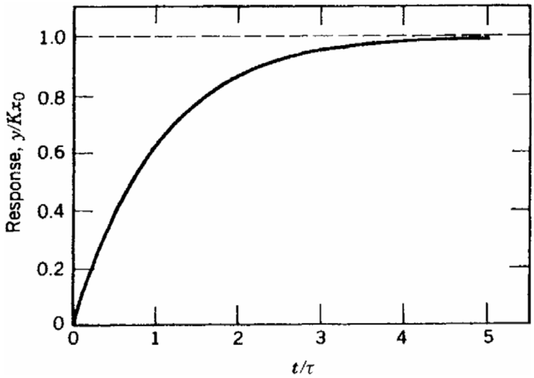

Figure F.4 Response of a first-order instrument to a step change input versus nondimensional time.

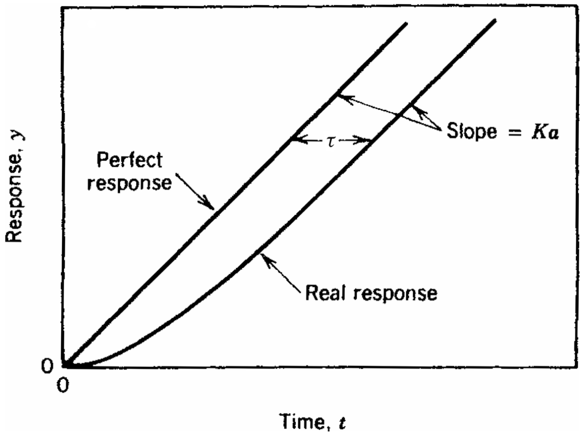

Figure F.5 Response of a first-order instrument to a ramp input versus time.

Equation (F.9) is plotted in Figure F.5. For no dynamic response error, we would obtain $y = Kat$ and the right-hand side (RHS) of Eq. (F.10) would be zero. The two terms on the RHS therefore represent the error in the response. The exponential term ($Ka\tau e^{-t/\tau}$) dies out with time and is called the *transient error*. The other term ($-Ka\tau$) is constant and proportional to τ. The smaller the time constant is, the smaller this steady-state error will be. The effect of the steady-state error is that the output does not correspond to the input at the current time but to the input τ seconds before.

The response of a first-order instrument to a sinusoidal input is found by solving (Eq. F.7) using Eq. (F.4) for x. This solution is

$$y = Ce^{-t/\tau} + \frac{KX}{\sqrt{1 + \omega^2\tau^2}} \sin(\omega t + \phi) \tag{F.11}$$

where

$$\phi = \tan^{-1}(-\omega\tau) \tag{F.12}$$

and C is the arbitrary constant of integration. The exponential term in Eq. (F.11) is the transient error that dies out in a few time constants. The second term on the RHS of Eq. (F.11) is the steady sinusoidal response of the instrument.

By comparing the steady response with the input [Eq. (F.4)], we see that the response has an amplitude error proportional to the amplitude coefficient $(1/\sqrt{1 + \omega^2\tau^2})$ and a phase error ϕ. These errors are shown in Figures F.6 and F.7. Each of these errors varies with the product of the time constant τ and the frequency of the input signal ω. As $\omega\tau$ increases, the amplitude coefficient decreases and the deviation from a perfect response becomes greater and greater, as seen in Figure F.6. A similar behavior is observed in Figure F.7 for the phase error, which asymptotically approaches -90° as $\omega\tau$ increases.

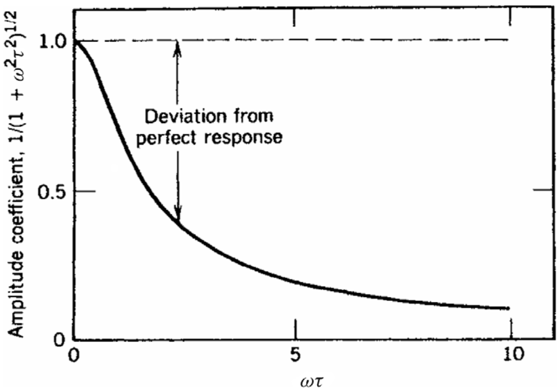

Figure F.6 Amplitude response of a first-order instrument to a sinusoidal input of frequency ω.

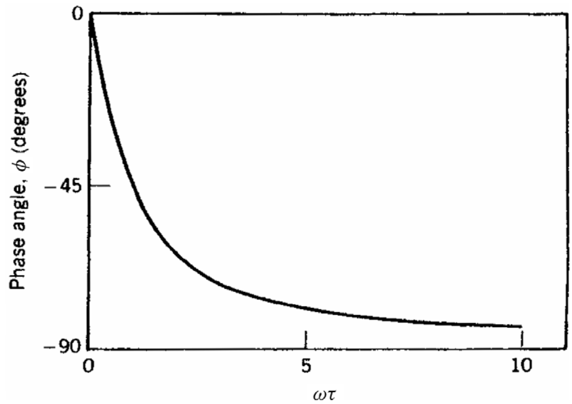

Figure F.7 Phase error in the response of a first-order instrument to a sinusoidal input of frequency ω.

F-4 RESPONSE OF SECOND-ORDER INSTRUMENTS

With $n = 2$ in Eq. (F.1), the response of a second-order instrument becomes

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bx(t) \tag{F.13}$$

If this expression is divided through by a_2, it can be written in the form

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = K\omega_n^2 x(t) \tag{F.14}$$

where $K(= b/a_0)$ is again the static gain, $\zeta(= a_1/2\sqrt{a_0a_2})$ is the damping factor, and $\omega_n(= \sqrt{a_0/a_2})$ is the natural frequency. The definition of a second-order instrument is one that has a dynamic response behavior that can be expressed in the form of Eq. (F.14) [3]. Instruments that exhibit a spring–mass type of behavior are second order. Examples are galvanometers, accelerometers, diaphragm-type pressure transducers, and U-tube manometers [1].

The nature of the solutions to Eq. (F.14) is determined by the value of the damping constant ζ. For $\zeta < 1$, the system is said to be *underdamped* and the solution is oscillatory. For $\zeta = 1$, the system is *critically damped*, and for $\zeta > 1$ the system is said to be *overdamped*.

The second-order instrument response to a step change is found by solving Eq. (F.14) using Eq. (F.2) for x and the initial conditions that $y = y' = 0$ at $t = 0$. The solution depends on the value of ζ and is given by:

$\zeta > 1$:

$$y = Kx_0\left\{1 - e^{-\zeta\omega_n t}\left[\cosh(\omega_n t\sqrt{\zeta^2 - 1}) + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sinh(\omega_n t\sqrt{\zeta^2 - 1})\right]\right\} \tag{F.15}$$

$\zeta = 1$:

$$y = Kx_0[1 - e^{-\omega_n t}(1 + \omega_n t)] \tag{F.16}$$

$\zeta < 1$:

$$y = Kx_0\left\{1 - e^{-\zeta\omega_n t}\left[\frac{1}{\sqrt{1 - \zeta^2}}\sin(\omega_n t\sqrt{1 - \zeta^2} + \phi)\right]\right\} \tag{F.17}$$

where

$$\phi = \sin^{-1}(\sqrt{1 - \zeta^2}) \tag{F.18}$$

This response is shown in Figure F.8. Note that $1/\zeta\omega_n$ is now the time constant. The larger $\zeta\omega_n$ is, the more quickly the response approaches the steady-state value. The form of the approach to the steady-state value is determined by ζ. For $\zeta < 1$, the response overshoots, then oscillates about the final value while being damped.

Most instruments are designed with damping factors of about 0.7. The reason for this can be seen in Figure F.8. If an overshoot of 5% is allowed, a damping factor $\zeta \simeq 0.7$ will result in a response that is within 5% of the steady-state value in about half the time required by an instrument with $\zeta = 1$ [1]. Note that the steady-state solution for all values of $\zeta > 0$ gives Kx_0.

The response to a ramp input also contains a transient and steady-state portion. The steady-state solution is

$$y = Ka\left(t - \frac{2\zeta}{\omega_n}\right) \tag{F.19}$$

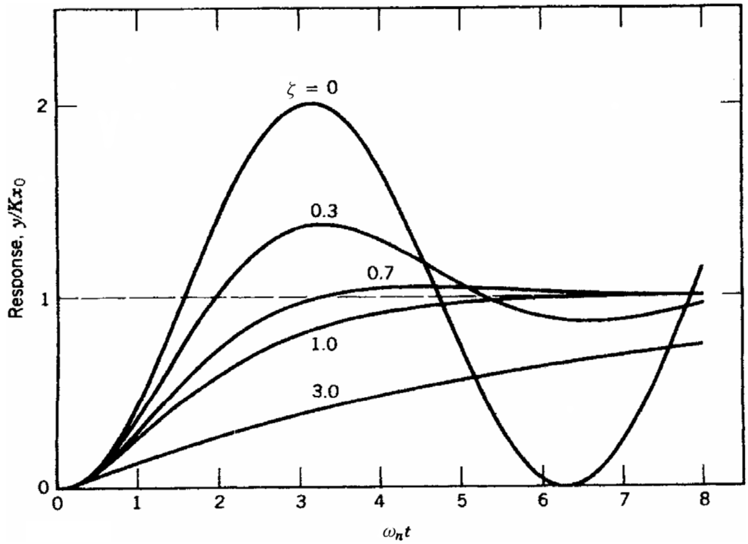

Figure F.8 Response of a second-order instrument to a step change input for various damping factors.

The response lags behind the input by a time equal to $2\zeta/\omega_n$. High values of ω_n and/or low values of ζ reduce this lag in the steady-state response.

The response of a second-order instrument to a sinusoidal input is (at steady state) given by

$$y = \frac{KX}{[(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2}} \sin(\omega t + \phi) \tag{F.20}$$

where

$$\phi = \tan^{-1}\left(-\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \tag{F.21}$$

As in the first-order system, the response contains both an amplitude error proportional to an amplitude coefficient and a phase error. These errors are shown in Figures F.9 and F.10.

From Figure F.9 we see that for no damping ($\zeta = 0$) the amplitude of the response approaches infinity as the input signal frequency ω approaches the instrument natural frequency ω_n. In general, this maximum amplitude, or resonance, will occur at

$$\omega = \omega_n\sqrt{1-2\zeta^2} \tag{F.22}$$

Note that for nonzero damping (which is always the case physically) the amplitude at this resonant frequency will be finite. Also note that for $\zeta \simeq 0.6$ to 0.7 and

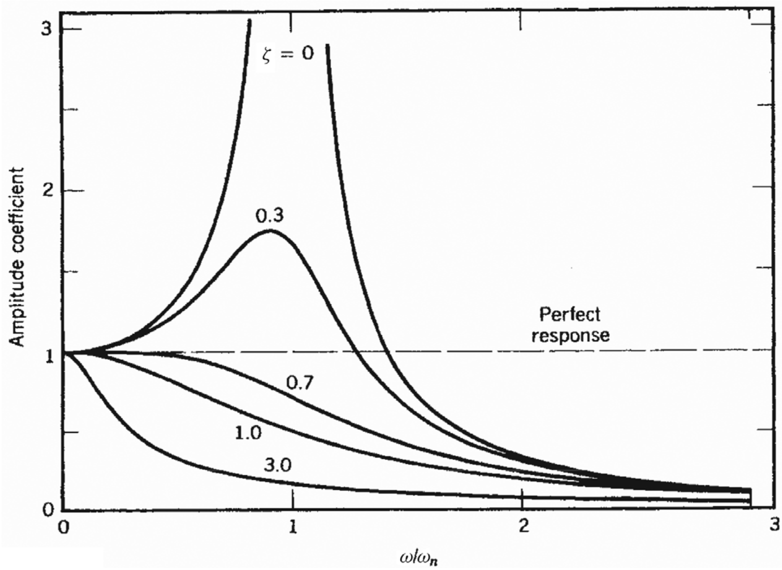

Figure F.9 Amplitude response of a second-order instrument to a sinusoidal input of frequency ω.

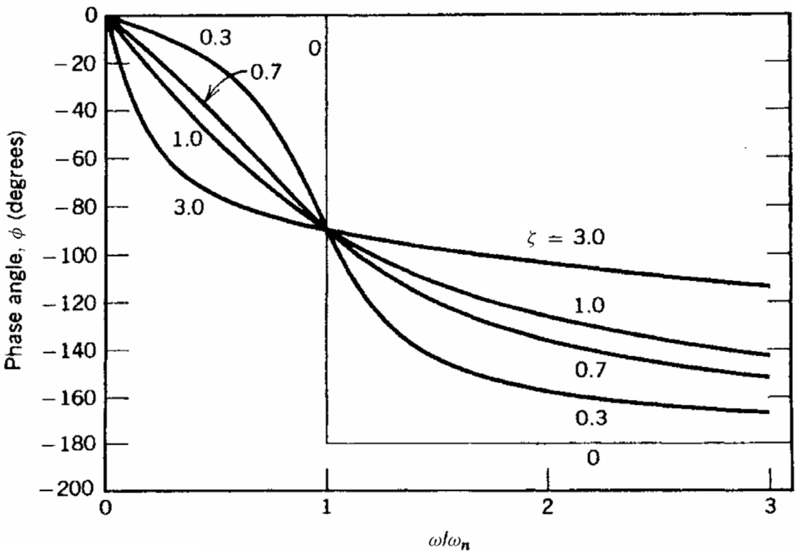

Figure F.10 Phase error in the response of a second-order instrument to a sinusoidal input of frequency ω.

$\omega/\omega_n < 1$, the amplitude error is minimized and the phase error is about linear, which is desirable because this produces minimum distortion of the input signal [1].

F-5 SUMMARY

The dynamic response of zero-, first-, and second-order instruments has been presented for step, ramp, and sinusoidal inputs. In the case of a zero-order instrument, it was found that there is no dynamic response error. For first- and second-order instruments, there are time delays for step and ramp inputs and amplitude and phase errors for sinusoidal inputs. By choosing or designing instruments with appropriate values of time constant and natural frequency, the effects of these errors can be minimized.

In a complete measurement system, different instruments will usually be connected in the form of a transducer, a signal conditioning device, and a readout. An example might be a thermocouple (a first-order instrument) connected to an analog voltmeter (a second-order instrument). In such cases the dynamic output of the first instrument can be determined and this value can be used as the input to the second instrument. The dynamic response of the second instrument to this input is then determined to obtain the dynamic response of the system. In most cases the significant dynamic response error effect will occur with only one of the instruments, usually the transducer.

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