MASTERING TECHNIQUES OF DIFFERENTIATION

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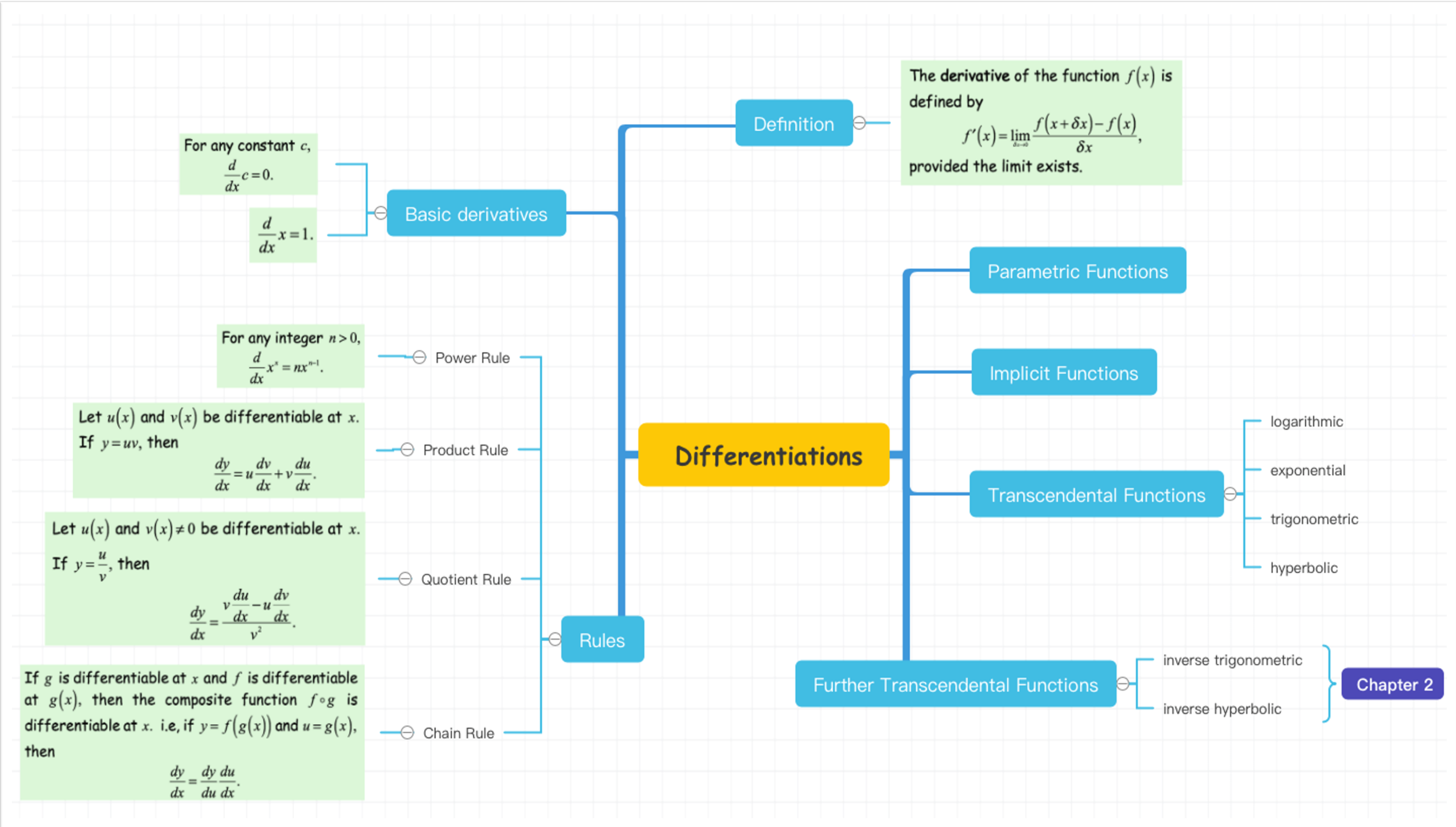
Differentiations
Definition
The derivative of the function $f(x)$ is defined by $f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$, provided the limit exists.
Basic derivatives
For any constant c, $\frac{d}{dx}c=0.$
$\frac{d}{dx}x=1.$
Rules
Power Rule
For any integer $n>0$, $\frac{d}{dx}x^n=nx^{n-1}.$
Product Rule
Let $u(x)$ and $v(x)$ be differentiable at x. If $y=uv$, then $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$
Quotient Rule
Let $u(x)$ and $v(x)\neq 0$ be differentiable at x. If $y=\frac{u}{v}$, then $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$
Chain Rule
If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f\circ g$ is differentiable at x. i.e, if $y=f(g(x))$ and $u=g(x)$, then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}.$
Parametric Functions
Implicit Functions
Transcendental Functions
logarithmic
exponential
trigonometric
hyperbolic
Further Transcendental Functions
inverse trigonometric
inverse hyperbolic
Chapter 2

1. THE DERIVATIVES

Differentiation refers to the process of computing a derivative. The derivative is associated with the concept of rate of change. The formal definition of derivative is given as follows:

> **Definition 1 : Derivative**
>
> The derivative of the function $f(x)$ is defined by
>
> $$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x},$$
>
> provided the limit exists.

It can be time-consuming and complicated to find derivatives of functions using the definition of the derivative. Simpler techniques include differentiation rules. The basic derivatives are presented first.

> For any constant c,
>
> $$\frac{d}{dx} c = 0.$$

Example 1.1

$$y = f(x) = 3.$$

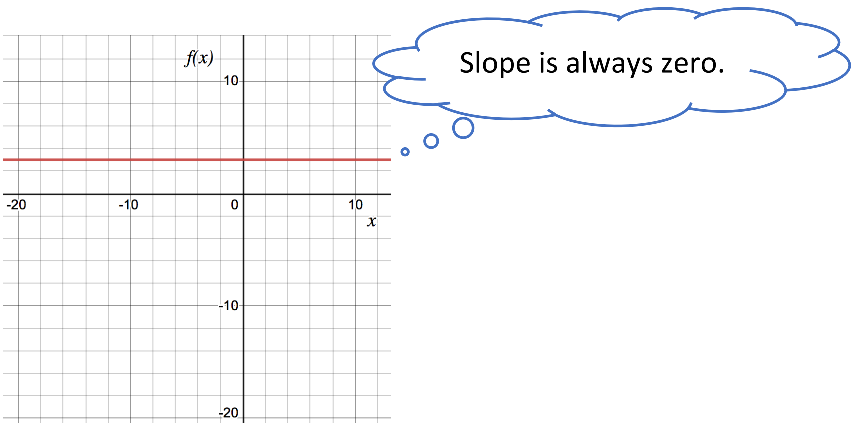

Figure 1.1 Graph of $f(x) = 3$

Let $f(x) = x.$

$$\frac{d}{dx}x = 1.$$

Example 1.2

Differentiate $y = f(x) = x$ with respect to x.

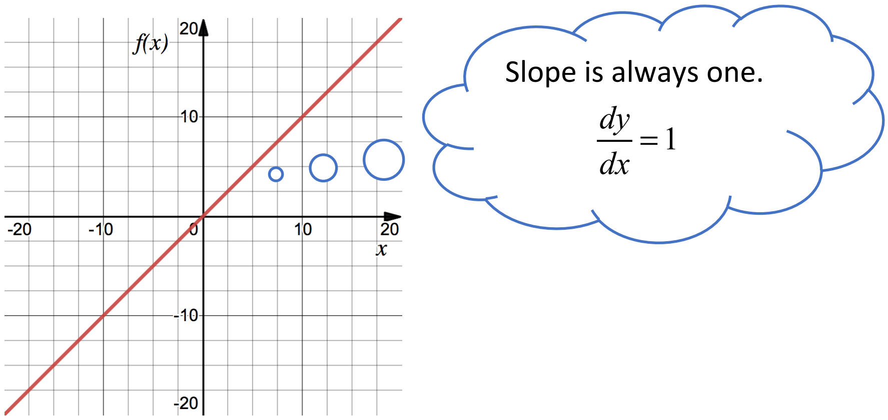

Figure 1.2 Graph of $f(x) = x$

The following rules for combining derivatives further expand the number of derivatives without resorting to the definition.

For any constant c, let $y = cu(x)$. If $u(x)$ is a differentiable function of x, then

$$\frac{dy}{dx} = c\frac{du}{dx}.$$

Example 1.3

Differentiate $y = 2x$ with respect to x.

Let $u(x)$ and $v(x)$ be differentiable functions of x. If $y = u \pm v$, then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Example 1.4

Differentiate $y = 5x + 3$ with respect to x.

2. RULES OF DERIVATIVES

Operations such as addition, subtraction, multiplication, division, composition, and inversion on several functions can result in the formation of a new function. Given that the original functions are differentiable, the new function obtained can also been shown to be differentiable and can be expressed in terms of component functions and their derivatives.

2.1 POWER RULE

Power Rule

For any integer $n > 0$,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

Example 2.1

Differentiate the following functions with respect to x:

i. $y=-\frac{3}{4}x^{12}$,

ii. $f\left(x\right)=\sqrt{6x}$.

Extended Power Rule

For any negative integer k,

$$\frac{d}{dx}x^{k}=kx^{k-1}.$$

Example 2.2

Differentiate the following functions with respect to x:

i. $y=x^{-6}$,

ii. $y=\frac{1}{x}$,

iii. $f\left(x\right)=\frac{3}{\sqrt[3]{x}}$.

2.2 PRODUCT RULE

Product Rule

Let $u(x)$ and $v(x)$ be differentiable at $x.$ If $y = uv,$ then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 2.3

Differentiate the following functions with respect to x:

i. $y = \left(2x^3 - x^2 + 5\right)\left(x^4 - 3x^2 + 2\right),$

ii. $y = \sqrt{x}\left(x - \frac{1}{x}\right).$

2.3 QUOTIENT RULE

Quotient Rule

Let $u(x)$ and $v(x) \neq 0$ be differentiable at x. If $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 2.4

Differentiate the following functions with respect to x:

i. $y = \frac{x+3}{x-1}$,

ii. $y = \frac{5}{2+\sqrt{x}}$.

2.4 CHAIN RULE

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x. i.e, if $y = f(g(x))$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 2.5

Differentiate the following functions with respect to x:

i. $f(x)=(3x^2+1)^5$,

ii. $f(x)=\sqrt{5x+2}$.

3. DIFFERENTIATION OF IMPLICIT FUNCTIONS

Some functions can be explicitly defined, for example $y = x^2 + 3$ (parabola). However, there are exists equations such as the equation of a circle $x^2 + y^2 = 1$. where y cannot be expressed in terms of x only. Such a relation is said to be implicitly defined and can be written in the form $F(x, y) = 0$ instead of $y = f(x)$. In implicit differentiation, both sides of the equation are differentiated by treating one of the variables as the function of the other.

Example 3.1

Find $\frac{dy}{dx}$ for each of the following functions:

i. $\frac{x}{y^3} = 3.$

ii. $(x + y)^2 = 2.$

4. DIFFERENTIATION OF PARAMETRIC FUNCTIONS

The relationships between two variables x and y can be defined in parametric form by using parameter t as $x = x(t)$ and $y = y(t)$. The derivative $\frac{dy}{dx}$ can be found as follows :

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}, \text{ where } \frac{dt}{dx} = \frac{1}{dx/dt}.$$

Example 4.1

Find $\frac{dy}{dx}$ for each of the following functions:

i. $x = 2t + 4, \;\; y = -8t,$

ii. $x = 3t^2 - 2, \;\; y = 4t^2 - 6.$

5. DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

A transcendental function is a function which is not an algebraic function. In this section, differentiations of some transcendental functions which are logarithmic functions, exponential functions and trigonometric functions will be discussed.

5.1 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential and logarithmic functions are among the most commonly encountered functions in applications such as business and economics. Basic derivatives for the exponential and logarithmic functions are given as follows:

$f(x)$	$f'(x)$
e^x	e^x
e^{kx} (k constant)	ke^{kx}
$\log x$ ($x>0$)	$\frac{1}{x}$
a^x (a constant)	$a^x \ln(a)$

Example 5.1

Find the derivative of the following functions:

i. $f(x)=e^{\frac{-x}{2}}$,

ii. $f(x)=x^2e^{-x}$,

iii. $f(x)=\frac{e^x}{2^x}$.

Example 5.2

Find the derivative of the following functions:

i. $f(x) = \ln x^4$,

ii. $f(x) = e^x \ln x$

5.2 TRIGONOMETRIC FUNCTIONS

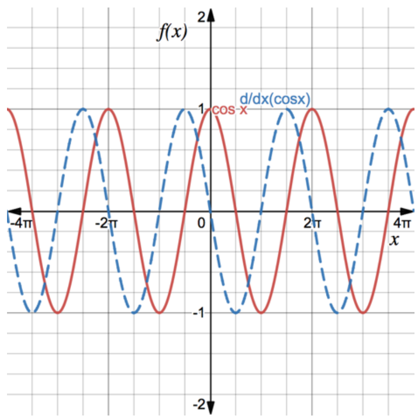

Figure 4.1 Graph of $f(x)=\cos x$ and $\frac{d}{dx}(\cos x)$

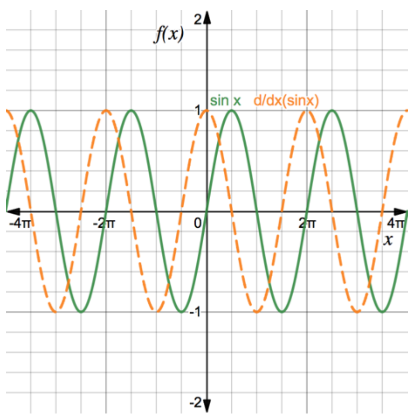

Figure 4.2 Graph of $f(x)=\sin x$ and $\frac{d}{dx}(\sin x)$

Derivatives of $\cos x$ and $\sin x$ as given in the following table can be deduced based on Definition 1.

y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Derivatives for other trigonometric functions can also be obtained by using the differentiation rules which have been discussed in Section 2.

Example 5.3

By using differentiation rules, prove the following:

i. $\frac{d}{dx}(\tan x)=\sec^2 x$,

ii. $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$,

iii. $\frac{d}{dx}(\sec x)=\sec x\tan x$,

iv. $\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$.

Example 5.4

Find $\frac{dy}{dx}$ for the following functions :

i. $y = \cos^2 x \sin 2x,$

ii. $y = e^{\sin 2x},$

iii. $y = \sec\left(7 + \pi^2 x\right)$

iv. $\cos\left(x + y\right) = 10 + y \cos x^2$