ONLINE CAMP MODULE

MASTERING TECHNIQUES OF INTEGRATION

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1. ANTIDERIVATIVES

By reversing the process of differentiation, the antiderivative of a function can be obtained. The definition of antiderivative is given as follows:

> **Definition 1 : Antiderivative**
>
> An **antiderivative** of a function $f(x)$ is a function whose derivative is $f(x)$. In other words, a function $F(x)$ is an **antiderivative** of a function $f(x)$ if
>
> $$F'(x) = f(x)$$
>
> for all x in the domain of f.

In general, for any constant C,

$$\frac{d}{dx}\left[F(x) + C\right] = F'(x) + 0 = f(x).$$

The following theorem shows that the antiderivative of a function $f(x)$ is unique up to adding a constant.

> **Theorem 1 :**
>
> Suppose that F and G are both antiderivatives of f in the interval $[a,b]$. Then,
>
> $$G(x) = F(x) + C,$$
>
> For some constant C.

The process of computing the antiderivative is called integration.

Definition 2 : Indefinite Integral

Let F be any antiderivative f. The indefinite integral of $f(x)$ is

$$\int f(x)\,dx = F(x) + C,$$

Integral symbol: $\int$; Integrand: $f(x)$; Variable of integration: dx; Constant of integration: C

where C is an arbitrary constant.

Note that $\frac{d}{dx}\left[f(x)\right] = f'(x)$ is equivalent to $\int f'(x)\,dx = f(x) + C$. Recall that for every rational power, n,

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1},$$

which implies that $\frac{d}{dx}\left[x^{n+1}\right] = (n+1)x^n$.

By reversing the process, the following rule is obtained.

Power Rule

For any rational power $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Adding the power by 1

Divide by the new power

Example 1.1

Evaluate the following :

i. $\int 3\,dx$

ii. $\int x^{21}dx$

iii. $\int \frac{dx}{x^2}$

iv. $\int \frac{1}{\sqrt[3]{x}}dx$

The power rule is only applicable for x^n with $n \neq -1$. For the case $n = -1$, we have $x^n = x^{-1} = \frac{1}{x}$.

> For $x \neq 0$,
>
> $$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Note that the above integration rule is obtained from the derivative

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}, \text{ for } x \neq 0.$$

By applying Chain Rule,

$$\frac{d}{dx}\left[\ln|f(x)|\right] = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}, \text{ for } f(x) \neq 0.$$

Thus, the following integration rule is obtained.

> For $f(x) \neq 0$,
>
> $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Example 1.2

i. $\int \frac{1}{x}\,dx$

ii. $\int \frac{1}{x+1} - \frac{1}{x^2}\,dx$

Corresponding Derivative Formula	Indefinite Integral
$\frac{d}{dx}\left[C\right]=0,\quad C$ constant	$\int 0\,dx=C,\ C$ constant
$\frac{d}{dx}\left[x\right]=1$	$\int dx=x+C$
$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right]=x^n,\ \ n\neq -1$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\quad n\neq -1$
$\frac{d}{dx}\left[\ln\lvert x\rvert\right]=\frac{1}{x}$	$\int \frac{dx}{x}=\ln\lvert x\rvert+C$
$\frac{d}{dx}\left[e^x\right]=e^x$	$\int e^x dx=e^x+C$

Algebraic Properties Indefinite Integrals

a) The constant factor k can be taken out from the integral i.e

$$\int kf(x)\,dx=k\int f(x)\,dx$$

b) Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then

$$\int\left[af(x)\pm bg(x)\right]dx=a\int f(x)dx\pm b\int g(x)dx,$$

for any constant a and $b.$

Example 1.2

Evaluate

i. $\int \sqrt{4x}-7x^5dx.$

ii. $\int e^x+\frac{2}{x}dx$

iii. $\int \frac{dx}{\sqrt[3]{27x}}$

The indefinite integral which has been discussed is integral without the limits of integration. Meanwhile, the integral with the limits of integration is known as definite integral.

Definition 3 : Definite Integral

The definite integral of a function $f(x)$ is of the form

Upper limit (constant)

Variable of integration

$$\int_a^b f(x)\,dx$$

Lower limit (constant)

Integrand

Fundamental Theorem of Calculus

If a function $f(x)$ is continuous on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is a function such that $F'(x) = f(x)$ for all $x \in [a,b]$.

Based on the Fundamental Theorem of Calculus, basic properties of definite integrals are given as follows :

Basic Properties of Definite Integrals

If $f(x)$ and $g(x)$ are continuous functions on the interval $[a,b]$,then

a) $\int_a^a f(x)\,dx = 0$ if $f(a)$ exists

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

d) $\int_a^b k\,dx = k(b-a)$

e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a \le c \le b$

f) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example 1.3

i. $\int_{1}^{3} 0\, dx$

ii. $\int_{1}^{4} 8x^{3} + 3\, dx$

2. TECHNIQUES OF INTEGRATION

2.1 Integration by Substitutions

Let $u = g(x)$ and $f(u)$ be a function in terms of u. Then,

$$\int \left[f(u) \frac{du}{dx} \right] dx = \int f(u)\, du.$$

Step 1 : Let $u = g(x)$.

Step 2 : Obtain $\frac{du}{dx} = g'(x)$.

Step 3 : Substitute $u = g(x)$ and $du = g'(x)\,dx$.

(After substitution, the whole integral must be in terms of u.)

Step 4 : Evaluate $\int f(u)\, du$.

Step 5 : Substitute back u with $g(x)$.

(Final answer must be in terms of x.)

Example 2.1

i. $\int \frac{2x}{x^2+1} dx$

ii. $\int \frac{dx}{(3-2x)^{1/3}}$

iii. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

iii. $\int_0^3 \frac{3x+2}{x-4} dx$

2.2 Integration by Parts

Integration by parts is the process of integrating a product of two functions by splitting up the integrand into two parts. Recall that, if u and v are functions of x, the product rule of differentiation gives us

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

which implies that

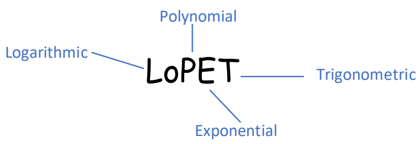
$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

By integrating both sides with respect to x, we obtain the formula for integration by parts as follows:

By parts

$$\int u\,dv = uv - \int v\,du$$

One of the parts, corresponding to u, will be differentiated and the other part, corresponding to dv, will be integrated. In making the choice of u and dv, it is advisable that, for the chosen u, $\frac{du}{dx}$ is simpler than u.



Example 2.2

i. $\int \ln x\,dx$

ii. $\int x^2 e^{-x}\,dx$

iii. $\int_1^2 x^2 \ln x\,dx$

2.3 Integration by Tabular Method

Integration by parts will become tedious when the power of the corresponding functions increases as repeated differentiations and integrations need to be done. In this case, tabular method, which is a special case of integration by parts.

Tabular method

$$\int uv'\,dx$$

Conditions:

a) u can be differentiated easily with respect to x until becoming zero

b) v' can be integrated with respect to x easily

OR

If the differentiation does not yield to zero, double differentiation-integration process produce terms which are multiple of u and v'.

Example 2.3

Evaluate $\int x^4 e^{-3x} dx.$

2.4 Integration of Rational Functions Using Partial Fractions

Partial fractions

Types of denominator	**Function**	**Partial fraction**
Linear factor	$\frac{f(x)}{x+a}$	$\frac{A}{x+a}$
Repeated linear factor	$\frac{f(x)}{(x+a)^2}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}$
	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}+\frac{C}{(x+a)^3}$
Quadratic factor	$\frac{f(x)}{x^2+px+q}$	$\frac{Ax+B}{x^2+px+q}$
Repeated quadratic factor	$\frac{f(x)}{(x^2+px+q)^2}$	$\frac{Ax+B}{x^2+px+q}+\frac{Cx+D}{(x^2+px+q)^2}$

Example 2.4

i. $\int \frac{9x-8}{x^2-x}dx.$

ii. $\int \frac{5x+1}{x^2-x-12}dx$

3. INTEGRATION OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

The integration formula of the trigonometric and hyperbolic functions obtained by reversing the corresponding derivative formula is listed in the following table:

Corresponding Derivative Formula	**Indefinite Integral**
$\frac{d}{dx}\left[\cos x\right]=-\sin x$	$\int \sin x\,dx=-\cos x+C$
$\frac{d}{dx}\left[\sin x\right]=\cos x$	$\int \cos x\,dx=\sin x+C$
$\frac{d}{dx}\left[\tan x\right]=\sec^2 x$	$\int \sec^2 x\,dx=\tan x+C$
$\frac{d}{dx}\left[\cot x\right]=-\text{cosec}^2 x$	$\int \text{cosec}^2 x\,dx=-\cot x+C$
$\frac{d}{dx}\left[\sec x\right]=\sec x\tan x$	$\int \sec x\tan x\,dx=\sec x+C$
$\frac{d}{dx}\left[\text{cosec}\,x\right]=-\text{cosec}\,x\cot x$	$\int \text{cosec}\,x\cot x\,dx=-\text{cosec}\,x+C$
$\frac{d}{dx}\left[\cosh x\right]=\sinh x$	$\int \sinh x\,dx=\cosh x+C$
$\frac{d}{dx}\left[\sinh x\right]=\cosh x$	$\int \cosh x\,dx=\sinh x+C$
$\frac{d}{dx}\left[\tanh x\right]=\text{sech}^2 x$	$\int \text{sech}^2 x\,dx=\tanh x+C$
$\frac{d}{dx}\left[\coth x\right]=-\text{cosech}^2 x$	$\int \text{cosech}^2 x\,dx=-\coth x+C$
$\frac{d}{dx}\left[\text{sech}\,x\right]=-\text{sech}\,x\tanh x$	$\int \text{sech}\,x\tanh x\,dx=-\text{sech}\,x+C$
$\frac{d}{dx}\left[\text{cosech}\,x\right]=-\text{cosech}\,x\coth x$	$\int \text{cosech}\,x\coth x\,dx=-\text{cosech}\,x+C$
$\frac{d}{dx}\left[\ln\left\|\sec x+\tan x\right\|\right]=\sec x$	$\int \sec x\,dx=\ln\left\|\sec x+\tan x\right\|+C$
$\frac{d}{dx}\left[\ln\left\|\text{cosec}\,x+\cot x\right\|\right]=-\text{cosec}\,x$	$\int \text{cosec}\,x\,dx=-\ln\left\|\text{cosec}\,x+\cot x\right\|+C$

Example 3.1

i. $\int \tan x \, dx$

ii. $\int (\sin x + \cos x)^2 \, dx$

iii. $\int_{-\frac{\pi}{2}}^{0} \cos^3 x \sin x \, dx$

Example 3.2

i. $\int \operatorname{sech}^2 x \tanh x \, dx$

ii. $\int \sin x \sinh 2x \, dx$

iii. $\int 3\cosh^2 5x \, dx$

4. INTEGRATION OF INVERSE TRIGONOMETRIC AND INVERSE HYPERBOLIC FUNCTIONS

Differentiations of Inverse Functions
$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx},\ \lvert u\rvert < 1.$
$\frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx},\ \lvert u\rvert < 1.$
$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2} \cdot \frac{du}{dx}.$
$\frac{d}{dx}[\cot^{-1} u] = \frac{-1}{1+u^2} \cdot \frac{du}{dx}.$
$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx},\ \lvert u\rvert > 1.$
$\frac{d}{dx}[\text{cosec}^{-1} u] = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx},\ \lvert u\rvert > 1.$
$\frac{d}{dx}[\sinh^{-1} u] = \frac{1}{\sqrt{u^2+1}} \cdot \frac{du}{dx}$
$\frac{d}{dx}[\cosh^{-1} u] = \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{dx},\ \lvert u\rvert > 1.$
$\frac{d}{dx}[\tanh^{-1} u] = \frac{1}{1-u^2} \cdot \frac{du}{dx},\ \lvert u\rvert < 1.$
$\frac{d}{dx}[\coth^{-1} u] = \frac{1}{1-u^2} \cdot \frac{du}{dx},\ \lvert u\rvert > 1.$
$\frac{d}{dx}[\text{sech}^{-1} u] = \frac{-1}{u\sqrt{1-u^2}} \cdot \frac{du}{dx},\ 0 < u < 1.$
$\frac{d}{dx}[\text{cosech}^{-1} u] = \frac{-1}{\lvert u\rvert\sqrt{1+u^2}} \cdot \frac{du}{dx},\ u \neq 0.$

Integrations Resulting in Inverse Functions
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$
$\int \frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C.$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C.$
$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C,\ a > 0.$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C,\ x > 0.$
$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C, & \lvert x\rvert < a, \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C, & \lvert x\rvert > a. \end{cases}$
$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\text{sech}^{-1}\left(\frac{x}{a}\right) + C,\ 0 < x < a.$
$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\text{cosech}^{-1}\left\lvert\frac{x}{a}\right\rvert + C,\ 0 < x < a.$

Example 4.1

i. $\int \sin^{-1} x dx$

ii. $\int \frac{\cosh^{-1} x}{\sqrt{x^2-1}} dx$

iii. $\int \frac{x-2}{x^2-4x+8} dx$