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General Zeroth-Order Randić Index of the Zero Divisor Graph for Some Commutative Rings

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Abstract. For a simple graph Γ with the set of edges and vertices, the general zeroth-order Randić index is defined as the sum of the degree of each vertex to the power of $\alpha \neq 0$. Meanwhile, the zero divisor graph of a ring is defined as a simple graph with vertex set is the set of zero divisors of the ring and a pair of distinct vertices a, b in the ring are adjacent if and only if $ab = 0$. This paper is an endeavor to construct the formula of the general zeroth-order Randić index of the zero divisor graph for some commutative rings. The commutative ring in the scope of this research is the ring of integers modulo p^n, where n is a positive integer and p is a prime number. The general zeroth-order Randić index is found for the cases $\alpha = 1, 2$ and 3.

INTRODUCTION

Researchers have examined topological indices initially only for graphs involving chemical structures, but then began to examine graphs in general. In addition, the concept of graphs obtained from algebraic structures is also becoming more and more established. Thus, topological indices of graphs can be calculated from algebraic structures such as rings. The nature and applications of topological indices have been studied by Basak et al. [1] as it has close relations to chemistry. Several topological indices exist, including the Wiener index, the Zagreb index, the Szeged index, the Harary index, the Degree-distance index, the Kirchoff index and more. One of them is Randić index which has been developed by Randić in 1975 [2] where the research on characterization of molecular branding has been carried out as well as chemistry and pharmacology make use of it the most. In particular, the design of quantitative structure-property (QSPR) and structure-activity relations (QSAR) [3]. In terms of topological indices, it is a degree-based index which usually be used by chemists. Then, the general zeroth-order Randić index has been developed by Li and Zheng [4]. The definition of the general zeroth-order Randić index is the sum of the degree of each vertex to the power of $\alpha \neq 0$.

By Elumalai and Mansour in [5], the expected value of the zeroth-order Randi's index has been quantified for all bargraphs with n cells. A general expression has been established for evaluating the zeroth-order general Randić index of line graphs of various well-known chemical structures. As a result of the research, Mehak and Bhatti in [6] are able to determine properties of physicochemical substances and biological processes of chemical structures in drugs. It has been proposed and studied by Anderson and Livingston in [7] that zero-divisor graphs can have nonzero zero divisors

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as their vertices. This graph is able to best illustrate the properties of a commutative ring's set of zero-divisors. A zero divisor graph allows us to use graph-theoretical tools to study the algebraic properties of rings. According to Seeta in [8], the zero divisor graph must be a star graph or a complete graph. The girth, diameter, and connectivity of the zero divisor graph have been examined. Smith in [9] restricted his attention to the zero divisor graph of the set of integers modulo n, $\mathbb{Z}_n$. The zero divisor graphs perfectly for all values of n that Smith determined.

This paper presents the theoretical results for constructing general formulas for the general zeroth-order Randić index of a zero divisor graph using the set of integers of modulo p^n. The general zeroth-order Randić index is found for the cases $\alpha = 1, 2$ and 3.

PRELIMINARIES

A brief review of the ring theory, graph theory and topological index are used in this research is presented in this section.

Definition 1 [10] Zero Divisor of a Ring
A nonzero element of a ring is said to be a zero divisor if the product of that nonzero element with another nonzero element of the ring is equal to zero.

The representation for the set of all zero divisors in a ring R is denoted by $Z(R)$ while the set of all nonzero zero divisors in R is represented by $Z(R)^*$.

Definition 2 [7] Zero Divisor Graph
Suppose $Z(R)^*$ is a set of all zero divisors of a ring R. The zero divisor graph of R, denoted by $\Gamma(R)$, is the vertex set consists of all zero divisors of R, and distinct vertices a, b are adjacent in this graph if and only if $ab = 0$.

Definition 3 [4] General Zeroth-order Randić Index
Let Γ be a connected graph and $deg(u)$ be the vertex degree u in the graph. Then,

$$R_\alpha^0 = \sum_{u \in V} \big(deg(u)\big)^\alpha$$

where α is an arbitrary real number.

Theorem 1 [9] Smith's Main Theorem
There are four forms in which the zero divisor graph of $\mathbb{Z}_n$ is perfect:
(a) $n = p^k$ for a prime p and positive integer k,
(b) $n = p^k q^l$ for distinct primes p, q and positive integers k, l,
(c) $n = p^k q^l r^m$ for distinct primes p, q, r and positive integers k, l, m,
(d) $n = pqrs$ for distinct primes p, q, r, s.

This research focuses on the first form of the Smith's main theorem, namely when $n = p^k$ for positive integer k and a prime p.

RESULTS AND DISCUSSION

This section presents some results related to the general zeroth-order Randić index of zero divisor graph for the ring of integers modulo p^n. As a first step, the following proposition determines a vertex's degree in the zero divisor graph of $\mathbb{Z}_{p^n}$.

Proposition 1 Let p be a prime number, $n \in \mathbb{N}$ and $a \in Z\big(\mathbb{Z}_{p^n}\big)^*$ with $gcd(a, p^n) = p^i$ for $i = 1,2, \dots, n$. Then, the degree of a in the zero divisor graph of $\mathbb{Z}_{p^n}$ is

$$deg(a) = \begin{cases} p^i - 1, & i \le \left\lfloor \frac{n-1}{2} \right\rfloor, \\ p^i - 2, & i > \left\lfloor \frac{n-1}{2} \right\rfloor. \end{cases}$$

where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p, and $\lfloor n \rfloor$ denotes the floor function of n.

Proof Let $a \in Z(\mathbb{Z}_{p^n})^*$ with $gcd(a, p^n) = p^i$ and $b \in Z(\mathbb{Z}_{p^n})^*$ with $gcd(b, p^n) = p^j$ for $i \ne j$. Then a is adjacent to b, $ab = 0$ if and only if $j \ge n - i$. So $|p^{n-i}\mathbb{Z}_{p^n}| = p^i - 1$. Thus, there are two cases where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p:

(1) If $i > \left\lfloor \frac{n-1}{2} \right\rfloor$, then $a \in p^{n-i}\mathbb{Z}_{p^n}$. So $deg(a) = p^i - 2$.

(2) If $i \le \left\lfloor \frac{n-1}{2} \right\rfloor$, then $deg(a) = p^i - 1$.

□

In the following proposition, the number of vertices that have the same degree is determined.

Proposition 2 Let $V'_i = \{a \in Z(\mathbb{Z}_{p^n})^* : gcd(a, p^n) = p^i\}$, then $|V'_i| = p^{n-i} - p^{n-(i+1)}$ for $1 \le i \le n-1$ where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p.

Proof Let $a \in V'_i$ for $1 \le i \le n-1$, we have $a \in p^i\mathbb{Z}_{p^n}$ but $a \notin p^{i+1}\mathbb{Z}_{p^n}$. So $|V'_i| = p^{n-i} - p^{n-(i+1)}$ where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p.

□

This proposition states the number of edges in the graph.

Proposition 3 Number of edges of $\Gamma(\mathbb{Z}_{p^n})$ is $\frac{1}{2}\left[(p^n - p^{n-1})(n-1) - p^{n-1} - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 2\right]$ where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p.

Proof In Handshaking lemma [11], any graph has half as many edges as the sum of the degrees of all its vertex are proven. Using Proposition 1 and Proposition 2, we have

$$\begin{aligned} |E(\Gamma\mathbb{Z}_{p^n})| &= \frac{1}{2} \sum_{a \in V(\Gamma\mathbb{Z}_{p^n})} deg(a) \\ &= \frac{1}{2}\left[\sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(p^{n-i} - p^{n-(i+1)}\right)(p^i - 1) + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(p^i - 2)\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n-1}(p^n - p^{n-i} - p^{n-1} + p^{n-i-1}) + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} (p^{n-i-1} - p^{n-i})\right] \\ &= \frac{1}{2}\left[(p^n - p^{n-1})(n-1) - p^{n-1} - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 2\right], \end{aligned}$$

where $n \ge 3$ for $p = 2$ and $n \ge 2$ for odd primes p.

□

Next, the main results in this research, for $\alpha = 1, 2$ and 3 are presented in the following theorems.

Theorem 2 The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 1$ is $(p^n - p^{n-1})(n-1) - p^{n-1} - p^{\left\lceil \frac{n-1}{2} \right\rceil} + 2$ where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

Proof By using Definition 3, Proposition 1 and Proposition 2, we have

$$
\begin{aligned}
R_1^0 &= \sum_{u \in V} \left(deg(u)\right)^1 \\
&= \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(p^{n-1} - p^{n-(i+1)}\right)(p^i - 1)^1 + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(p^i - 2)^1 \\
&= \sum_{i=1}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(p^i - 1) - p^n \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} \left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right) \\
&= (p^n - p^{n-1})(n-1) - p^{\left\lceil \frac{n-1}{2} \right\rceil} - p^{n-1} + 2,
\end{aligned}
$$

where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

□

Theorem 3 The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 2$ is $2(p^{n-1} - p^n)\left(n - 1 - \left\lceil \frac{n-1}{2} \right\rceil\right) + p^{2n-1} + p^{n-1} - p^n + 3\left(p^{\left\lceil \frac{n-1}{2} \right\rceil} - 1\right) - 1$ where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

Proof By using Definition 3, Proposition 1 and Proposition 2, we have

$$
\begin{aligned}
R_2^0 &= \sum_{u \in V} \left(deg(u)\right)^2 \\
&= \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(p^{n-1} - p^{n-(i+1)}\right)(p^i - 1)^2 + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(p^i - 2)^2 \\
&= \sum_{i=1}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(p^{2i} - p^i + 1) + \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1} \left(p^{n-i} - p^{n-(i+1)}\right)(3 - 2p^i) \\
&= 2\sum_{i=1}^{n-1}(p^{n-1} - p^n) + p^n \sum_{i=1}^{n-1}\left(p^i + \frac{1}{p^i} - p^{i-1} - \frac{1}{p^{i+1}}\right) + 2\sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1}(p^{n-1} - p^n) + 3p^n \sum_{i=1+\left\lfloor \frac{n-1}{2} \right\rfloor}^{n-1}\left(\frac{1}{p^i} - \frac{1}{p^{i+1}}\right) \\
&= 2(p^{n-1} - p^n)\left(n - 1 - \left\lceil \frac{n-1}{2} \right\rceil\right) + p^{2n-1} + p^{n-1} - p^n + 3\left(p^{\left\lceil \frac{n-1}{2} \right\rceil} - 1\right) - 1,
\end{aligned}
$$

where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

□

Theorem 4 The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^n}$ when $\alpha = 3$ is $\frac{p^{3n-1} - p^{n+1}}{p+1} + 3(p^n - p^{n-1})\left(n - 1 + 3\left\lceil \frac{n-1}{2} \right\rceil\right) - 6p^{2n-1} + 3p^n - p^{n-1} + 3p^{\left\lfloor \frac{3n-1}{2} \right\rfloor} - 7p^{\left\lfloor \frac{n}{2} \right\rfloor} + 8$ where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

Proof By using Definition 3, Proposition 1 and Proposition 2, we have

$$
\begin{aligned}
R_3^0 &= \sum_{u\in V}\big(deg(u)\big)^3 \\
&= \sum_{i=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\big(p^{n-1}-p^{n-(i+1)}\big)\,(p^i-1)^3 + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1}\big(p^{n-i}-p^{n-(i+1)}\big)\,(p^i-2)^3 \\
&= \sum_{i=1}^{n-1}\big(p^{n-i}-p^{n-(i+1)}\big)\,(p^{3i}-3p^{2i}+3p^i-1) + \sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1}\big(p^{n-i}-p^{n-(i+1)}\big)(p^{3i}-6p^{2i}+12p^i-8) \\
&= (p^n-p^{n-1})\left[\sum_{i=1}^{n-1}\left(p^{2i}-3p^i+3-\frac{1}{p^i}\right)+\sum_{i=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n-1}\left(9-3p^i-\frac{1}{p^i}\right)\right] \\
&= \frac{p^{3n-1}-p^{n+1}}{p+1}+3(p^n-p^{n-1})\left(n-1+3\left\lfloor\frac{n-1}{2}\right\rfloor\right)-6p^{2n-1}+3p^n-p^{n-1}+3p^{\left\lfloor\frac{3n-1}{2}\right\rfloor}-7p^{\left\lfloor\frac{n}{2}\right\rfloor}+8,
\end{aligned}
$$

where $n \geq 3$ for $p = 2$ and $n \geq 2$ for odd primes p.

□

CONCLUSION

Some results related to the general zeroth-order Randić index of the zero divisor graph for the ring of integers modulo p^n are found in this paper. These include the degree of a vertex, the number of vertices and edges in the zero divisor graph of $\mathbb{Z}_{p^n}$. Finally, the general formulas of the general zeroth-order Randić index of the zero divisor graph for the ring of integers modulo p^n have been constructed for the cases $\alpha = 1, 2$ and 3. Other general formulas for computing the general zeroth-order Randić index for the ring of integers modulo can also be computed.

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REFERENCES

1. S.C. Basak, V.R. Magnuson, G.J. Niemi, R.R. Regal, and G.D. Veith, “Topological indices: their nature, mutual relatedness, and applications,” Mathematical Modelling **8**, 300–305, (1987).
2. M. Randic, “Characterization of molecular branching,” J. Am. Chem. Soc. **97**(23), 6609–6615, (1975).
3. I. Gutman, B. Furtula, and V. Katanić, “Randić index and information,” AKCE International Journal of Graphs and Combinatorics **15**(3), 307–312, (2018).
4. X. Li, and J. Zheng, “A unified approach to the extremal trees for different indices,” MATCH Commun. Math. Comput. Chem **54**(1), 195–208, (2005).
5. S. Elumalai, and T. Mansour, “On the general zeroth-order Randic index of bargraphs,” Discrete Mathematics Letters **2**, 6–9, (2019).
6. G.E. Mehak, and A.H. Bhatti, “Zeroth-order general Randić index of line graphs of some chemical structures in drugs,” Scientific Bulletin-" Politehnica" University of Bucharest. Series B, Chemistry and Materials Science (Online) **81**(1), 47–70, (2019).
7. D.F. Anderson, and P.S. Livingston, “The zero-divisor graph of a commutative ring,” J Algebra **217**(2), 434–447, (1999).
8. V. Seeta, “Zero divisor graph of a commutative ring.,” Turkish Online Journal of Qualitative Inquiry **12**(7), (2021).

9. B. Smith, “Perfect zero-divisor graphs of $\mathbb{Z}_n$,” Rose-Hulman Undergraduate Mathematics Journal **17**(2), 6, (2016).
10. J.B. Fraleigh, *A First Course in Abstract Algebra* (Pearson Education India, 2003).
11. D.S. Gunderson, *Handbook of Mathematical Induction: Theory and Applications* (CRC Press, 2014).