On the Vertices of the Zero Divisor Graph of Some Finite Ring of Matrices of Dimension Two

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Abstract. A graph is a mathematical subject that consists of vertices and edges. Many studies have been done on the graphs of algebraic structures, including groups and rings. A zero divisor graph of a finite ring is defined as a graph with all zero divisors of the ring as its vertices, where two vertices are adjacent when the product of the vertices is the zero element of the ring. In this paper, we discuss on the exact number of vertices of the zero divisor graph of some finite ring of matrices. The results are found by using the properties of the determinant of a matrix, and given in the form of general formula.

INTRODUCTION

A ring is an algebraic structure defined as a set that is an Abelian group by addition, associative by multiplication, and also satisfies the distributive law [1]. Rings have uncountable interesting features that spark the interest of many algebraists to investigate further, such as their units, ideals, annihilators, and zero divisors. Due to their interesting features, rings have also been applied to many other areas of mathematics, including probability theory, cryptography, as well as graph theory. In this paper, a finite graph associated with rings, namely the zero divisor graph, is studied on a type of noncommutative ring, which is the ring of matrices of dimension two. First, the definition of the zero divisors of a finite ring is given as follows:

Definition 1 Zero Divisors of A Finite Ring [1]
If a and b are two nonzero elements of a ring R such that $ab = 0$ where 0 is the zero element of the ring R, then a and b are the zero divisors of R.

The set of all zero divisors of a ring R is denoted as $Z(R)$. Next, the definition of a zero divisor graph of a noncommutative ring is provided in the following.

Definition 2 Zero Divisor Graph of A Finite Noncommutative Ring [2]
Let R be a noncommutative ring. The zero divisor graph of R, $\Gamma(R)$ is a simple and directed graph with the zero divisors $Z(R)$ as its vertices, and an arc $x \to y$ is formed between distinct vertices x and y if and only if $xy = 0$.

In other words, the zero divisor graph visualizes the zero product property of a ring by showing the relations between two zero divisors of the ring.

This paper is divided into five sections. The first section is the introduction, followed by the literature review on the advancement of the zero divisor graph in the second section. The third section presents the research methodology. Then, the fourth section provides the main results of this study. Finally, the conclusion of the study is provided in the fifth section.

THE ADVANCEMENT OF THE ZERO DIVISOR GRAPH OF FINITE RINGS

In this section, the development of the zero divisor graph throughout the years is presented. The idea of the zero divisor graph of finite rings was first given by Beck [3] in 1988. The author studied the coloring of graph vertices, where the graph in the subject was a graph where all elements of a ring R are its vertices and two vertices are adjacent if and only if their product is the zero element of R.

Later on in 1993, Anderson and Nasser [4] studied the results given by Beck in [3] and provided a counterexample of Beck's results on graph coloring. However, a formal definition of the graph was not yet provided. Ultimately, Anderson and Livingston [5] provided the definition of the graph in 1999, and named the graph as the zero divisor graph. One of the fundamental characteristics of the zero divisor graph of a commutative ring found by the authors [5] is the graph must be a connected graph. The definition of the zero divisor graph is given as follows:

Definition 3 Zero Divisor Graph of a Commutative Ring [5]
Let R be a commutative ring and let $Z(R)$ be its set of zero divisors. The zero divisor graph of R, $\Gamma(R)$ is a simple and undirected graph with the elements of $Z(R)$ as its vertices, and for distinct $x, y \in Z(R)$, the vertices x and y are adjacent if and only if $xy = yx = 0$.

Subsequently, numerous studies have been conducted on the zero divisor graph of commutative rings. This includes the studies done on the Von Neumann regular rings [6], reduced rings [7], and the ring of all real-valued continuous functions on a completely regular Hausdorff space [8].

Eventually, Redmond [2] was the first who extended the notion of the zero divisor graph on finite noncommutative rings. The definition of the zero divisor graph defined by Redmond [2] has been given previously in Definition 2. In the study, Redmond [2] found that the zero divisor graph, $\Gamma(R)$ of a finite noncommutative ring is a directed graph. In addition, it was also determined that in finite noncommutative rings, $\Gamma(R)$ need not be a connected graph.

The study of the zero divisor graph has also resulted in many extensions. For example, in 2011, Afkhami and Khashyarmanesh [9] introduced the cozero-divisor graph of a commutative ring, where the vertices are narrowed down to only non-unit elements of a ring. Besides that, Spiroff and Wickham [10] in 2011 studied the graph of equivalence classes of zero divisors of a commutative Noetherian ring with unity, R. The vertices of the graph are the classes of elements in $Z(R)$, and each pair of distinct classes $[x], [y]$ is joined by an edge if and only if $[x] \cdot [y] = [0]$.

In addition, Bennis et al. [11] defined another extension of the zero divisor graph, which is the extended zero divisor graph. The vertices of this graph are the zero divisors of a finite ring R and two distinct vertices, say x and y, are adjacent if and only if there exist two nonnegative integers n and m such that $x^n y^m = 0$ where $x^n \neq 0$ and $y^m \neq 0$.

Moreover, the quasi-zero divisor graph was introduced by Zhao et al. [12] in 2017 where the vertices of the graph are the quasi-zero divisors of a finite ring R and two vertices x and y are adjacent if and only if $xy = 0$. An element $x \in R$ is said to be a quasi-zero divisor if there exists $0 \neq y \in R$ such that $xy = 0$.

Later on, Cherrabi et al. [13] introduced another extension to the zero divisor graph, denoted as $\widetilde{\Gamma}(R)$, where the vertices are all zero divisors of a finite ring R and two vertices x and y are adjacent if and only if $xy = 0$ or $x + y \in Z(R)$. Based on the definition, it can be seen that the original zero divisor graph $\Gamma(R)$ defined by Anderson and Livingston [5] is a subgraph of this extended graph since the edges of $\Gamma(R)$ are certainly the edges of $\widetilde{\Gamma}(R)$.

In 2020, Li, Miller and Tucci [14] extended the zero divisor graph to a graph where loops can be created on the graph's vertices. By allowing loops, the properties of the graph including the degree of vertices and their neighborhood elements changed. It is concluded that if the zero divisor graphs (with loops) of two finite commutative rings have the same number of vertices, edges, and loops, then the finite commutative rings are isomorphic. In 2022, Bajaj and Panigrahi [15] officially named the graph as the looped zero divisor graph and the graph was studied on some finite commutative rings $\mathbb{Z}_n$, where their focus was to obtain the graph's universal adjacency spectrum.

In 2021, Zai et al. [16] introduced the complement of the zero divisor graph based on Beck's definition in [12]. The graph is named the non-zero divisor graph of a finite ring R. The vertices of the graph are all non-zero elements of R and two distinct elements x and y in R are adjacent if and only if $xy \neq 0$.

In this paper, we used the definition of the zero divisor graph given by Redmond in [2] to obtain the exact number of vertices of the zero divisor graph of a type of noncommutative ring, specifically the ring of matrices of dimension two over integers modulo prime. The next section presents the methodology for obtaining the main results of this study.

RESEARCH METHODOLOGY

In this section, the method of determining the main results of this study is discussed. Since this study aims to determine the number of vertices of the zero divisor graph, hence based on Definition 2, we need to determine the number of the zero divisors of the ring. The ring in the focus of this study is the ring of 2×2 matrices over integers modulo p, where p is prime. The following proposition given by McCoy [17] depicts a method of determining the zero divisors of the ring of matrices, by using the determinant of a matrix.

Proposition 1 [17] Let A be a given element of the ring of $n \times n$ matrices, M_n with elements in the commutative ring R. Then A is a zero divisor of M_n if and only if the determinant of A is a divisor of zero in R.

In other words, a matrix A in a ring of matrices M_n over integers modulo m is a zero divisor if and only if its determinant is zero (mod m). Subsequently, some basic properties of a matrix with zero determinant are given in the following theorems.

Theorem 1 [18] In a square matrix A, if all elements of one of the rows or columns are zero, then the determinant of A, $det(A) = 0$.

Theorem 2 [18] If two parallel rows or columns of a square matrix A are equal, then $det(A) = 0$.

Theorem 3 [18] If two parallel lines of a square matrix A are proportional, then $det(A) = 0$.

In the next section, the number of vertices of the zero divisor graph in this study is found based on Proposition 1 and given in the form of a general formula.

MAIN RESULTS

In this section, the computations of the main results, which is the number of vertices of the zero divisor graph, is given for the ring of 2×2 matrices over integers modulo p, $M_2(\mathbb{Z}_p)$, where p is any prime number. The number of vertices of the zero divisor graph of $M_2(\mathbb{Z}_p)$, which is also the number of zero divisors of $M_2(\mathbb{Z}_p)$, is obtained by determining the number of matrices in $M_2(\mathbb{Z}_p)$ that has determinant zero based on Proposition 1.

Proposition 2 Let R be a ring of matrices of dimension two over integers modulo p, where p is prime and $Z(R)$ is its set of zero divisors. Then, $|Z(R)| = p^3 + p^2 - p - 1$.

Proof. Suppose R is a ring of matrices of dimension two over integers modulo p, $\mathbb{Z}_p$ where p is prime. Based on Proposition 1, an element A of a ring of matrices R is a zero divisor of R if the determinant of A is zero.

To determine the number of zero divisors in R, the number of element $A \in R$ where the determinant of A is nonzero is firstly determined. Let $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. In the first column of A, there are $p^2 - 1$ possible choices of elements in $\mathbb{Z}_p$. The column where $x_1, x_2 = 0$ is excluded as it automatically yields zero determinant for the matrix A. Then, for the second column, there are $p^2 - p$ possible choices, because matrices with two proportional columns have determinant zero, based on Theorem 3.

Hence, the number of matrices with nonzero determinant is $(p^2 - 1)(p^2 - p) = p^4 - p^3 - p^2 + p$. Then, the number of matrices in R with zero determinant is $p^4 - (p^4 - p^3 - p^2 + p) - 1 = p^3 + p^2 - p - 1$; where p^4 is the order of the ring of matrices R and 1 is subtracted from the total because the zero matrix is excluded from the calculation. Hence, the number of zero divisors of the ring R, $|Z(R)| = p^3 + p^2 - p - 1$. □

The following example presents the number of zero divisors of the ring of matrices of dimension two over integers modulo p when $p = 3$.

Example 1 Given a finite ring $R = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{Z}_3 \right\}$. Based on Definition 1, it is found that the set of the zero divisors of R,

$$Z(R)=\left\{\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&2\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}0&0\\1&2\end{pmatrix},\begin{pmatrix}0&0\\2&0\end{pmatrix},\begin{pmatrix}0&0\\2&1\end{pmatrix},\begin{pmatrix}0&0\\2&2\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\right.$$
$$\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&2\end{pmatrix},\begin{pmatrix}0&2\\0&0\end{pmatrix},\begin{pmatrix}0&2\\0&1\end{pmatrix},\begin{pmatrix}0&2\\0&2\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\2&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},$$
$$\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\2&2\end{pmatrix},\begin{pmatrix}1&2\\0&0\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix},\begin{pmatrix}2&0\\0&0\end{pmatrix},\begin{pmatrix}2&0\\1&0\end{pmatrix},\begin{pmatrix}2&0\\2&0\end{pmatrix},\begin{pmatrix}2&1\\0&0\end{pmatrix},$$
$$\left.\begin{pmatrix}2&1\\1&2\end{pmatrix},\begin{pmatrix}2&1\\2&1\end{pmatrix},\begin{pmatrix}2&2\\0&0\end{pmatrix},\begin{pmatrix}2&2\\1&1\end{pmatrix},\begin{pmatrix}2&2\\2&2\end{pmatrix}\right\}.$$

Therefore, $|Z(R)| = 32$. The result is consistent with Proposition 2, where the number of zero divisors of R, $|Z(R)| = 3^3+3^2-3-1 = 32$. □

Next, based on Proposition 2, the following corollary is obtained.

Corollary 1 Let $M_2(\mathbb{Z}_p)$ be a ring of matrices of dimension two over integers modulo p. Then, the zero divisor graph of $M_2(\mathbb{Z}_p)$, $\Gamma(M_2(\mathbb{Z}_p))$ has p^3+p^2-p-1 vertices.

Proof. Based on Definition 2, the vertices of a zero divisor graph of a noncommutative ring are the zero divisors of the ring. Then, according to Proposition 2, the ring of matrices of dimension two over integers modulo p, $M_2(\mathbb{Z}_p)$ has p^3+p^2-p-1 zero divisors. Therefore, the number of vertices of its zero divisor graph, $\Gamma(M_2(\mathbb{Z}_p))$ is p^3+p^2-p-1.

CONCLUSION

In this paper, a general formula is established for computing the number of zero divisors of the finite ring of 2×2 matrices over integers modulo p, where p is prime. To formulate the general formula, the properties of the determinant of square matrices are utilized. It is found that the number of vertices of the ring is based on the value of p.

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