J. Appl. Math. & Informatics Vol. 42(2024), No. 3, pp. 663 - 680
https://doi.org/10.14317/jami.2024.663

ON THE TOPOLOGICAL INDICES OF ZERO DIVISOR GRAPHS OF SOME COMMUTATIVE RINGS†

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Abstract. The zero divisor graph is the most basic way of representing an algebraic structure as a graph. For any commutative ring R, each element is a vertex on the zero divisor graph and two vertices are defined as adjacent if and only if the product of those vertices equals zero. In this research, we determine some topological indices such as the Wiener index, the edge-Wiener index, the hyper-Wiener index, the Harary index, the first Zagreb index, the second Zagreb index, and the Gutman index of zero divisor graph of integers modulo prime power and its direct product.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : Wiener index, hyper-wiener index, Harary index, edge-Wiener index, first Zagreb index, second Zagreb index, Gutman index, zero divisor graph.

1. Introduction

In chemistry, the graph theory has been widely used to solve molecular problems. The structure of molecules can be represented as a graph where atoms are vertices, and the bonds between atoms are edges. Some can be assumed as a complete molecular graph, while others can be viewed as a skeleton graph [1].

There are many applications of graph theory and group theory in chemistry. One of them is topological indices that represent the chemical structure with numerical values. Additionally, the topological index of a structure is useful for chemical documentation, isomer discrimination, structure-property relationships, and others [2].

For instance, the Wiener index (W) is used to predict the cavity surface area (CSA) of alcohols with the equation $ln(csa) = 5.229 + 0.144ln(W)$, predict the

Received November 19, 2023. Revised January 21, 2024. Accepted January 24, 2024.
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†This research was funded by ITB International Research Grant 2021.
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boiling point (BP) of alcohols with the equation $ln(bp) = 4.279 + 0.181ln(W)$, and predict the molar refraction (MR) of heterogeneous compounds with the equation $ln(mr) = 0.826 + 0.690ln(W)$ (Gupta, [3]). Many types of topological indices have been introduced since 1947, which include the Wiener index [4, 5] and the Zagreb index [8, 7]. The present study investigated the structure of these indices in a specific graph, namely the zero divisor graph.

The zero divisor graph is the most basic way of representing an algebraic structure as a graph. Here, for any commutative ring R, each element is a vertex on the zero divisor graph and two vertices are defined as adjacent if and only if the product of those vertices equals 0 [10, 9]. Previous studies refer to the zero divisor graph [11, 12] as a useful way of working with algebraic graphs, for example to manipulate each element in R or to find the relation between two elements.

The simplest ring (or commutative ring) that can be thought of as a fundamental structure is no other than $\mathbb{Z}_p$, or the modulo ring over a prime number. This modulo ring can be approached using elementary number theory to discover many modulo ring properties and relations. Any commutative ring can be represented by a modulo ring or the multiplication of some modulo rings. Based on this, the zero divisor graph as a complete representation of any commutative ring is interesting to explore [13, 14].

Rayer and Jeyaraj illustrated the zero divisor graph of commutative rings in 2023. They also examined topological indices for zero divisor graphs, focusing on the eccentricity of the vertices [15]. In the same year, Ghazali et al discovered general zeroth-order Randić index of zero divisor graph for the ring of integers modulo p^n [6].

In this paper, we present the general formula of the Wiener index, hyper-Wiener index, Harary index, edge-Wiener index, first Zagreb index, second Zagreb index and Gutman index of a zero divisor graph using the modulo ring of a prime power $\mathbb{Z}_{p^n}$ and its direct product $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ for prime numbers p, q and natural numbers n, m.

The definitions and basic concepts in graph theory and topological indices that are used to prove the main theorem are presented.
The zero divisor graph is defined in the following definition.

Definition 1.1. [10] Let R be a commutative ring. The zero divisor graph of R, denoted by ΓR, is a simple graph of the vertex set R and two distinct vertices x and y are joined by an edge whenever $xy = 0$.

Definition 1.2. [4] Let G be a connected graph. The Wiener index of G is the sum of the half of the distances between every unordered pair of vertices of G, written as,

$$W(G) = \sum_{u,v \in V(G)} d(u, v),$$

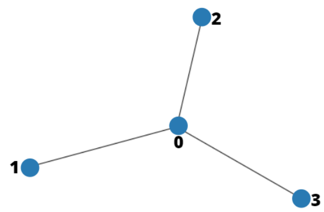

FIGURE 1. Zero divisor graph of $\mathbb{Z}_{2^2}$

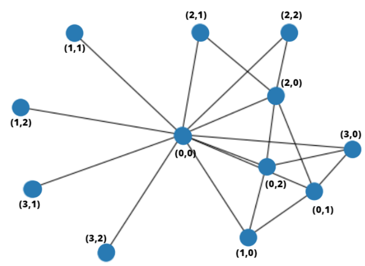

FIGURE 2. Zero divisor graph of $\mathbb{Z}_{2^2} \times \mathbb{Z}_3$

where $d(u, v)$ is the distances of unordered pair of vertices u and v.

Example 1.3. $W(\Gamma\mathbb{Z}_{2^2}) = d(0,1)+d(0,2)+d(0,3)+d(1,2)+d(1,3)+d(2,3) = 1+1+1+2+2+2 = 9.$

Definition 1.4. [16] Let G be a connected graph. The hyper-Wiener index of G, denoted by $WW(G)$, is defined as

$$WW(G) = \frac{1}{2}\bigg(\sum_{u,v \in V(G)} d(u,v) + \sum_{u,v \in V(G)} d(u,v)^2\bigg),$$

where $d(u, v)$ is the distances of unordered pair of vertices u and v.

Example 1.5. $WW(\Gamma\mathbb{Z}_{2^2}) = \frac{1}{2}\Big[d(0,1) + d(0,2) + d(0,3) + d(1,2) + d(1,3) + d(2,3) + d(0,1)^2 + d(0,2)^2 + d(0,3)^2 + d(1,2)^2 + d(1,3)^2 + d(2,3)^2\Big]$
$= \frac{1}{2}(1+1+1+2+2+2+1+1+1+4+4+4) = 12.$

Definition 1.6. [16] Let G be a connected graph. The Harary index of G, denoted by $H(G)$, is defined as

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)},$$

where $d(u,v)$ is the distances of unordered pair of vertices u and v.

Example 1.7. $H(\Gamma\mathbb{Z}_{2^2}) = \frac{1}{d(0,1)} + \frac{1}{d(0,2)} + \frac{1}{d(0,3)} + \frac{1}{d(1,2)} + \frac{1}{d(1,3)} + \frac{1}{d(2,3)} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{9}{2}$.

Definition 1.8. [17] Let G be a connected graph. The edge-Wiener index of G is the sum of the distances in the line graph between all pairs of edges of G, written as,

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e,f),$$

where $E(G)$ is the set of edges in G and $d(e,f)$ are the distances between two edges. The distances between two edges are the distances between the corresponding vertices in the line graph of G, denoted by $L(G)$.

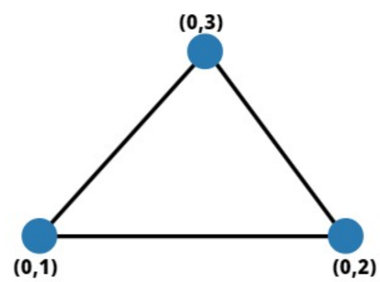

FIGURE 3. Zero divisor graph of line graph $\mathbb{Z}_{2^2}$

Example 1.9. $W_e(\Gamma\mathbb{Z}_{2^2}) = d\big((0,1),(0,2)\big) + d\big((0,1),(0,3)\big) + d\big((0,2),(0,3)\big) = 1 + 1 + 1 = 3$

Definition 1.10. [7] Let G be a connected graph. Then, the first Zagreb index of G is the sum of squares of the degrees of the vertices of G, written as,

$$M_1(G) = \sum_{u \in V(G)} deg(u)^2$$

where $deg(u)$ is the number of edges connected to vertex u.

Example 1.11. $M_1(\Gamma\mathbb{Z}_{2^2}) = deg(0)^2 + deg(1)^2 + deg(2)^2 + deg(3)^2 = 3^2 + 1^2 + 1^2 + 1^2 = 12$.

Definition 1.12. [7] Let G be a connected graph. Then, the second Zagreb index of G is the sum of the product of the degrees of pairs of adjacent vertices of G, written as,

$$M_2(G) = \sum_{u,v \in E(G)} deg(u)deg(v)$$

where u, v are the vertices on the edge connecting them.

Example 1.13. $M_2(\Gamma\mathbb{Z}_{2^2}) = deg(0)deg(1) + deg(0)deg(2) + deg(0)deg(3) = 3.1 + 3.1 + 3.1 = 9.$

Definition 1.14. [18] Let G be a connected graph. The Gutman index of G, denoted by $Gut(G)$, written as,

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} deg(u)deg(v)d(u,v)$$

.

Example 1.15. $Gut(\Gamma\mathbb{Z}_{2^2}) = deg(0)deg(1)d(0,1)+deg(0)deg(2)d(0,2)+deg(0)deg(3)d(0,3)+deg(1)deg(2)d(1,2)+deg(1)deg(3)d(1,3)+deg(2)deg(3)d(2,3) = 3.1.1 + 3.1.1 + 3.1.1 + 1.1.2 + 1.1.2 + 1.1.2 = 15.$

2. Main results

In this section, the Wiener index, the hyper-Wiener index, the Harary index, the edge-Wiener index, the first Zagreb index, and the second Zagreb index of the zero divisor graph of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ are determined, where their general forms in natural number n and and prime number p are found.

2.1. Topological indices of the graph that has diameter at most 2.

This subsection contains the Wiener index, the hyper-Wiener index, the Harary index, edge-Wiener index and Gutman index of graph that has diamater 2.

In terms of their numbers of vertices and number of edges, these results are used to prove the topological indices of zero divisor graphs of the commutative rings in the following subsections.

Theorem 2.1. *Let G be a simple connected graph with $diam(G) \leq 2$, then the Wiener index of G is $|V(G)|(|V(G)| - 1) - |E(G)|$.*

Proof. Since $\mathrm{diam}(G) \leq 2$, the number of unordered pairs of vertices in G that have distance 2 is,

$$\binom{|V(G)|}{2} - |E(G)|.$$

Hence, the Wiener index of G is,

$$W(G) = |E(G)| + 2\left(\binom{|V(G)|}{2} - |E(G)|\right)$$

$$= |V(G)|(|V(G)| - 1) - |E(G)|.$$

□

Theorem 2.2. *Let G be a simple connected graph with diam$(G) \leq 2$, then the hyper-Wiener index of G is $\frac{3}{2}|V(G)|(|V(G)| - 1) - 2|E(G)|$.*

Proof. Since $\text{diam}(G) \leq 2$, the number of unordered pairs of vertices in G that have distance 2 is,

$$\binom{|V(G)|}{2} - |E(G)|.$$

Hence, the hyper-Wiener index of G is,

$$\begin{aligned} WW(G) &= \frac{1}{2}\left[|E(G)| + 2\left(\binom{|V(G)|}{2} - |E(G)|\right) + |E(G)| + 4\left(\binom{|V(G)|}{2} - |E(G)|\right)\right] \\ &= \frac{3}{2}|V(G)|(|V(G)| - 1) - 2|E(G)|. \end{aligned}$$

□

Theorem 2.3. *Let G be a simple connected graph with diam$(G) \leq 2$, then the Harary index of G is $\frac{1}{4}|V(G)|(|V(G)| - 1) + \frac{1}{2}|E(G)|$.*

Proof. Since $\text{diam}(G) \leq 2$, the number of unordered pairs of vertices in G that have distance 2 is,

$$\binom{|V(G)|}{2} - |E(G)|.$$

Hence, the Harary index of G is,

$$\begin{aligned} H(G) &= |E(G)| + \frac{1}{2}\left(\binom{|V(G)|}{2} - |E(G)|\right) \\ &= \frac{1}{4}|V(G)|(|V(G)| - 1) + \frac{1}{2}|E(G)|. \end{aligned}$$

□

If diameter of the line graph is at most 2, then its edge-Wiener index in terms of number of edges and its first Zagreb index. This following lemma shows that.

Theorem 2.4. *Let G be a graph. If line graph of G is connected and has diameter at most 2, then $W_e(G) = |E(G)|^2 - \frac{1}{2}M_1(G)$.*

Proof. Two edges e, f in a graph G will be adjacent in $L(G)$ if both e and f share one same vertex in G. So every vertex v in G will have $deg(v)$ edges sharing the vertex v and create $\binom{deg(v)}{2}$ different edges in $L(G)$. Hence, we have

$$|E(L(G))| = \sum_{v \in V(G)} \binom{deg(v)}{2}$$

$$= \frac{1}{2} \sum_{v \in V(G)} deg(v)(deg(v) - 1)$$
$$= \frac{1}{2} \sum_{v \in V(G)} deg(v)^2 - \frac{1}{2} \sum_{v \in V(G)} deg(v)$$
$$= \frac{1}{2} M_1(G) - |E(G)|$$

By using Lemma 2.1, we have

$$\begin{aligned} W_e(G)) &= W(L(G)) \\ &= |V(L(G))| (|V(L(G))| - 1) - |E(L(G))| \\ &= |E(G)| (|E(G)| - 1) - \frac{1}{2} M_1(G) + |E(G)| \\ &= |E(G)|^2 - \frac{1}{2} M_1(G). \end{aligned}$$

□

If the graph has diameter at most 2, then, there is relationship between its Gutman index and its Zagreb indices. It shows in the following lemma.

Theorem 2.5. *Let G be a simple connected graph with $diam(G) \leq 2$, then the Gutman index of G is $Gut(G) = 4|E(G)|^2 - M_1(G) - M_2(G)$.*

Proof. Note that,

$$\begin{aligned} 4|E(G)|^2 &= \left(\sum_{u \in V(G)} deg(u) \right)^2 \\ &= \sum_{u \in V(G)} deg(u)^2 + 2 \sum_{uv \in E(G)} deg(u)deg(v) + 2 \sum_{uv \notin E(G)} deg(u)deg(v) \\ &= M_1(G) + 2M_2(G) + 2 \sum_{uv \notin E(G)} deg(u)deg(v). \end{aligned}$$

.

Since $diam(G) \leq 2$, then,

$$\begin{aligned} Gut(G) &= \sum_{uv \in E(G)} deg(u)deg(v)d(u,v) + \sum_{uv \notin E(G)} deg(u)deg(v)d(u,v) \\ &= \sum_{uv \in E(G)} deg(u)deg(v) + 2 \sum_{uv \notin E(G)} deg(u)deg(v) \\ &= M_2(G) + 4|E(G)|^2 - M_1(G) - 2M_2(G) \\ &= 4|E(G)|^2 - M_1(G) - M_2(G). \end{aligned}$$

□

2.2. Topological Indices of the Zero Divisor Graph of $\mathbb{Z}_{p^n}$. .
This subsection contains the results for the Wiener index, the hyper-Wiener index, the Harary index, the edge-Wiener index, the first Zagreb index, the second Zagreb index and Gutman index for $\Gamma\mathbb{Z}_{p^n}$ for prime number p and $n \in \mathbb{N}$.

Before getting into a deep understanding of how to determine any topological index result, the first thing to know is the graph's structure. The neighbors of each vertex are fundamental for seeing the structure preserved in that graph.

Lemma 2.6. *If R is a commutative ring, then the diameter of ΓR is at most 2.*

Proof. Let a, b be different vertices in ΓR with $ab \neq 0$. Since $a.0 = 0$ and $b.0 = 0$, $a - 0 - b$ is a path of length 2. □

Proposition 2.7. *Let p be a prime number, $n \in \mathbb{N}$ and $a \in \mathbb{Z}_{p^n}$ with $gcd(a, p^n) = p^i$ for $i = 0, 1, 2, ..., n$. Then, the degree of a in $\Gamma\mathbb{Z}_{p^n}$ is*

$$deg(a) = \begin{cases} p^i, & i \leq \lfloor \frac{n-1}{2} \rfloor \\ p^i - 1, & i > \lfloor \frac{n-1}{2} \rfloor \end{cases}$$

Proof. Let $a \in \mathbb{Z}_{p^n}$ with $\gcd(a, p^n) = p^i$, and $b \in \mathbb{Z}_{p^n}$ with $\gcd(b, p^n) = p^j$. Then, a is adjacent to b if and only if $j \geq n - i$. So $b \in p^{n-i}\mathbb{Z}_{p^n}$ and $|p^{n-i}\mathbb{Z}_{p^n}| = p^i$.

- If $i > \lfloor \frac{n-1}{2} \rfloor$, then $a \in p^{n-i}\mathbb{Z}_{p^n}$. So $deg(a) = p^i - 1$.
- If $i \leq \lfloor \frac{n-1}{2} \rfloor$, then $deg(a) = p^i$.

□

In the zero divisor graph of any commutative ring, some of the vertices have the same degree. The following proposition shows the number of vertices that have the same degree in the respective graphs.

Proposition 2.8. *Let $V_i = \{a \in \mathbb{Z}_{p^n} : gcd(a, p^n) = p^i\}$, then $|V_i| = p^{n-i} - p^{n-(i+1)}$ for $0 \leq i \leq n-1$ and $|V_i| = 1$ for $i = n$.*

Proof. Let $a \in V_i'$. For $0 \leq i \leq n-1$, we have $a \in p^i Z_{p^n}$, but $a \notin p^{i+1} Z_{p^n}$. So $|V_i'| = p^{n-i} - p^{n-(i+1)}$. For the case $i = n$, $V_i' = \{0\}$ and immediately $|V_i'| = 1$. □

Before we determine the Wiener index, Hyper-Wiener index, Harary index, edge-Wiener index and Gutman index, we need to determine the number of edges. The number of edges shows in the following lemma.

Lemma 2.9. *The number of edges of $\Gamma\mathbb{Z}_{p^n}$ is $\frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right)$.*

Proof. The number of edges in any graph will equal half of the sum of the degrees of all vertices. Using Proposition 2.7 and Proposition 2.8, we have,

$$|E(\Gamma\mathbb{Z}_{p^n})| = \frac{1}{2} \sum_{a \in V(\Gamma\mathbb{Z}_{p^n})} deg(a)$$

$$= \frac{1}{2}\left(p^n - 1 + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (p^{n-i} - p^{n-(i+1)})p^i + \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} (p^{n-i} - p^{n-(i+1)})(p^i - 1)\right)$$
$$= \frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right).$$

□

Theorem 2.10. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the Wiener index of $\Gamma\mathbb{Z}_{p^n}$ is*

$$W(\Gamma\mathbb{Z}_{p^n}) = p^n(p^n - 1) - \frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right).$$

Proof. Using Lemma 2.9 and Theorem 2.1 that state $|V(\Gamma\mathbb{Z}_{p^n})| = p^n$ and $|E(\Gamma\mathbb{Z}_{p^n})| = \frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right)$, the result follows. □

Example 2.11. $W(\Gamma\mathbb{Z}_{2^2}) = 2^2(2^2 - 1) - \frac{1}{2}\left((2+1)2^2 - 2.2^{2-1} - 2^{\lceil \frac{2-1}{2} \rceil}\right) = 9.$

Theorem 2.12. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the hyper-Wiener index of $\Gamma\mathbb{Z}_{p^n}$ is*

$$WW(\Gamma\mathbb{Z}_{p^n}) = \frac{3}{2}p^n(p^n - 1) - \left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right).$$

Proof. Using Lemma 2.9 and Theorem 2.2 that state $|V(\Gamma\mathbb{Z}_{p^n})| = p^n$ and $|E(\Gamma\mathbb{Z}_{p^n})| = \frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right)$, the result follows. □

Example 2.13. $WW(\Gamma\mathbb{Z}_{2^2}) = \frac{3}{2}.2^2(2^2 - 1) - \left((2+1)2^2 - 2.2^{2-1} - 2^{\lceil \frac{2-1}{2} \rceil}\right) = 12.$

Theorem 2.14. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the Harary index of $\Gamma\mathbb{Z}_{p^n}$ is*

$$H(\Gamma\mathbb{Z}_{p^n}) = \frac{1}{4}p^n(p^n - 1) + \frac{1}{4}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right).$$

Proof. Using Lemma 2.9 and Theorem 2.3 that state $|V(\Gamma\mathbb{Z}_{p^n})| = p^n$ and $|E(\Gamma\mathbb{Z}_{p^n})| = \frac{1}{2}\left((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil}\right)$, the result follows. □

Example 2.15. $H(\Gamma\mathbb{Z}_{2^2}) = \frac{1}{4}.2^2(2^2-1) + \frac{1}{4}\left((2+1)2^2 - 2.2^{2-1} - 2^{\lceil \frac{2-1}{2} \rceil}\right) = \frac{9}{2}.$

Before we determine the edge-Wiener index of $\Gamma\mathbb{Z}_{p^n}$, we need to determine the first Zagreb index.

Theorem 2.16. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the first Zagreb index of $\Gamma\mathbb{Z}_{p^n}$ is* $M_1(\Gamma\mathbb{Z}_{p^n}) = (p^n - 1)^2 + (p^n - p^{n-1})\left(\frac{(p^n-1)}{p-1} - 2\lceil \frac{n-1}{2} \rceil\right) + p^{\lceil \frac{n-1}{2} \rceil} - 1.$

Proof. By Proposition 2.7 and Proposition 2.8, we have,

$$M_1(\Gamma\mathbb{Z}_{p^n}) = (p^n-1)^2 + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})(p^i)^2 + \sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^{n-i}-p^{n-(i+1)})(p^i-1)^2.$$

$$= (p^n-1)^2 + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})(p^{2i}) + \sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^{n-i}-p^{n-(i+1)})(p^{2i}-2p^i+1).$$

$$= (p^n-1)^2 + (p^n-p^{n-1})\left(\frac{(p^n-1)}{p-1} - 2\lceil\tfrac{n-1}{2}\rceil\right) + p^{\lceil\frac{n-1}{2}\rceil} - 1.$$ □

Example 2.17. $M_1(\Gamma\mathbb{Z}_{2^2}) = (2^2-1)^2+(2^2-2^{2-1})\left(\frac{(2^2-1)}{2-1} - 2\lceil\frac{2-1}{2}\rceil\right)+2^{\lceil\frac{2-1}{2}\rceil} - 1 = 12.$

Theorem 2.18. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the edge-Wiener index of $\Gamma\mathbb{Z}_{p^n}$ is* $W_e(\Gamma\mathbb{Z}_{p^n}) = \frac{1}{4}\left((n+1)p^n - np^{n-1} - p^{\lceil\frac{n-1}{2}\rceil}\right)^2 - \frac{1}{2}\left((p^n-1)^2 + (p^n-p^{n-1})\left(\frac{p^n-1}{p-1} - 2\lceil\frac{n-1}{2}\rceil\right) + p^{\lceil\frac{n-1}{2}\rceil} - 1\right).$

Proof. For any prime numbers p and $n \geq 1$, let $a, b, c, d \in \mathbb{Z}_{p^n}$ be all different vertices in $\Gamma\mathbb{Z}_{p^n}$ such that $ab = cd = 0$. Then, $(a, b), (c, d) \in E(\Gamma\mathbb{Z}_{p^n})$ and it follows that $(a, b), (c, d)$ are different vertices in $L(\Gamma\mathbb{Z}_{p^n})$.
Let $0 \leq i, j, k, l \leq n$ be integers that satisfy $\gcd(a, p^n) = p^i$, $\gcd(b, p^n) = p^j$, $\gcd(c, p^n) = p^k$, and $\gcd(d, p^n) = p^l$. Note that $i + j \geq n$ and $k + l \geq n$, so $\max\{i, j\} + \max\{k, l\} \geq n$. Choose $x \in \{a, b\}$ and $y \in \{c, d\}$ that satisfy $\gcd(x, p^n) = p^{\max\{i,j\}}$ and $\gcd(y, p^n) = p^{\max\{k,l\}}$. Hence, $(x, y) \in E(\Gamma\mathbb{Z}_{p^n})$ and $(a, b) - (x, y) - (c, d)$ is a path of length 2. So $\mathrm{diam}(L(\Gamma\mathbb{Z}_{p^n})) \leq 2$.
Using Theorem 2.4, we have,

$$W_e(\Gamma\mathbb{Z}_{p^n})) = |E(\Gamma\mathbb{Z}_{p^n})|^2 - \frac{1}{2}M_1(\Gamma\mathbb{Z}_{p^n})$$

Using Lemma 2.9 and Theorem 2.16, the result follows. □

Example 2.19. $W_e(\Gamma\mathbb{Z}_{2^2}) = \frac{1}{4}\left((2+1)2^2 - 2.2^{2-1} - 2^{\lceil\frac{2-1}{2}\rceil}\right)^2$
$- \frac{1}{2}\left((2^2-1)^2 + (2^2-2^{2-1})\left(\frac{2^2-1}{2-1} - 2\lceil\frac{2-1}{2}\rceil\right) + 2^{\lceil\frac{2-1}{2}\rceil} - 1\right) = 3.$

Theorem 2.20. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the second Zagreb index of $\mathbb{Z}_{p^n}$ is*
$M_2(\mathbb{Z}_{p^n}) = (p^n-1)\left(n(p^n-p^{n-1}) - p^{\lceil\frac{n-1}{2}\rceil} + 1\right) + \frac{1}{2}(p^{\lceil\frac{n-1}{2}\rceil)} - 1)(p^{\lceil\frac{n-1}{2}\rceil} - 2)$
$+\frac{1}{2}(p^n-p^{n-1})\left(\frac{n(n-1)}{2}(p^n-p^{n-1}) + 2n + 2\lceil\frac{n-1}{2}\rceil - 1 - \frac{p^n-p+p^{\lceil\frac{n-1}{2}\rceil}-1}{p-1} - np^{\lceil\frac{n-1}{2}\rceil}\right).$

Proof. By the second Zagreb index's definition, it is obvious that

$$M_2(\Gamma G) = \frac{1}{2} \sum_{a \in V(\Gamma G)} \left(deg(a) \sum_{b \in N(a)} deg(b) \right),$$

where $N(a)$ denotes the neighborhood of a.
Note that, if $a \in V(\Gamma \mathbb{Z}_{p^n})$ with $gcd(a, p^n) = p^i$, then $N(a) = \{b \in V(\Gamma \mathbb{Z}_{p^n}) : gcd(b, p^n) = p^{n-j}, 0 \leq j \leq i\}$. Hence we have,

$$
\begin{aligned}
&M_2(\Gamma \mathbb{Z}_{p^n}) = \tfrac{1}{2} \left(\sum_{a \in V(\Gamma \mathbb{Z}_{p^n})} deg(a) \left(\sum_{b \in N(a)} deg(b) \right) \right) \\
&= \tfrac{1}{2}(p^n - 1) \Big((n+1)p^n - np^{n-1} - p^{\lceil \frac{n-1}{2} \rceil} - (p^n - 1) \Big) \\
&+ \tfrac{1}{2} \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (p^{n-i} - p^{n-(i+1)}) p^i \left(\sum_{j=1}^{i} (p^j - p^{j-1})(p^{n-j} - 1) + p^n - 1 \right) \\
&+ \tfrac{1}{2} \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} (p^{n-i} - p^{n-(i+1)})(p^i - 1) \Bigg(\sum_{j=1}^{\lceil \frac{n-1}{2} \rceil} (p^j - p^{j-1})(p^{n-j} - 1) \\
&+ \sum_{j=1+\lceil \frac{n-1}{2} \rceil}^{i} (p^j - p^{j-1})(p^{n-j}) + p^n - 1 - (p^i - 1) \Bigg) + \tfrac{1}{2}(p^n - p^{n-1})(p^n - 1) \\
&= \tfrac{1}{2}(p^n - 1) \Big(n(p^n - p^{n-1}) - p^{\lceil \frac{n-1}{2} \rceil} + 1 \Big) \\
&+ \tfrac{1}{2} \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (p^{n-i} - p^{n-(i+1)}) p^i + \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} (p^{n-i} - p^{n-(i+1)})(p^i - 1) \right) (p^n - 1) \\
&+ \tfrac{1}{2} \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (p^n - p^{n-1}) \big(i(p^n - p^{n-1}) - p^i + 1 \big) \\
&+ \tfrac{1}{2} \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} \big((p^n - p^{n-1}) - (p^{n-i} - p^{n-(i+1)}) \big) \Big(i(p^n - p^{n-1}) + 1 - p^{\lceil \frac{n-1}{2} \rceil} - p^i + 1) \Big) \\
&= \tfrac{1}{2}(p^n - 1) \Big(n(p^n - p^{n-1}) - p^{\lceil \frac{n-1}{2} \rceil} + 1 \Big) + \tfrac{1}{2} \Big(n(p^n - p^{n-1}) - p^{\lceil \frac{n-1}{2} \rceil} + 1 \Big) (p^n - 1) \\
&+ \tfrac{1}{2} \sum_{i=1}^{n-1} (p^n - p^{n-1}) \big(i(p^n - p^{n-1}) - p^i + 1 \big) + \tfrac{1}{2} \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} (p^n - p^{n-1}) \Big(1 - p^{\lceil \frac{n-1}{2} \rceil} \Big) \\
&- \tfrac{1}{2} \sum_{i=1+\lfloor \frac{n-1}{2} \rfloor}^{n-1} (p^{n-i} - p^{n-(i+1)}) \Big(i(p^n - p^{n-1}) + 1 - p^{\lceil \frac{n-1}{2} \rceil} - p^i + 1) \Big) \\
&= (p^n - 1) \Big(n(p^n - p^{n-1}) - p^{\lceil \frac{n-1}{2} \rceil} + 1 \Big) + \tfrac{1}{2}(p^n - p^{n-1}) \left(\tfrac{n(n-1)}{2}(p^n - p^{n-1}) - \tfrac{p^n - p}{p-1} + n - 1 \right) \\
&+ \tfrac{1}{2} \lceil \tfrac{n-1}{2} \rceil (p^n - p^{n-1}) \Big(1 - p^{\lceil \frac{n-1}{2} \rceil} \Big) + \tfrac{1}{2}(p^n - p^{n-1}) \left(n - \frac{p^{\lceil \frac{n-1}{2} \rceil} - 1}{p-1} - (1 + \lfloor \tfrac{n-1}{2} \rfloor) p^{\lceil \frac{n-1}{2} \rceil} \right) \\
&+ \tfrac{1}{2}(p^n - p^{n-1}) \lceil \tfrac{n-1}{2} \rceil + \tfrac{1}{2}(p^{\lceil \frac{n-1}{2} \rceil} - 1)(p^{\lceil \frac{n-1}{2} \rceil} - 2) \\
&= (p^n - 1) \Big(n(p^n - p^{n-1}) - p^{\lceil \frac{n-1}{2} \rceil} + 1 \Big) + \tfrac{1}{2}(p^{\lceil \frac{n-1}{2} \rceil)} - 1)(p^{\lceil \frac{n-1}{2} \rceil} - 2) \\
&+ \tfrac{1}{2}(p^n - p^{n-1}) \left(\tfrac{n(n-1)}{2}(p^n - p^{n-1}) + 2n + 2\lceil \tfrac{n-1}{2} \rceil - 1 - \frac{p^n - p + p^{\lceil \frac{n-1}{2} \rceil} - 1}{p-1} - np^{\lceil \frac{n-1}{2} \rceil} \right). \quad \square
\end{aligned}
$$

Example 2.21. $M_2(\Gamma\mathbb{Z}_{2^2}) = (2^2-1)\left(2(2^2-2^{2-1}) - 2^{\lceil\frac{2-1}{2}\rceil}+1\right) + \frac{1}{2}(2^{\lceil\frac{2-1}{2}\rceil)}-1)(2^{\lceil\frac{2-1}{2}\rceil}-2) + \frac{1}{2}(2^2-2^{2-1})\Big(\frac{2(2-1)}{2}(2^2-2^{2-1})+2.2+2\lceil\frac{2-1}{2}\rceil-1-\frac{2^2-2+2^{\lceil\frac{2-1}{2}\rceil}-1}{2-1}-2.2^{\lceil\frac{2-1}{2}\rceil}\Big) = 3.3+\frac{1}{2}.1.0+\frac{1}{2}(2+4+2-1-3-4) = 9.$

Theorem 2.22. *Let $\Gamma\mathbb{Z}_{p^n}$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ with prime number p and natural number n. Then, the Gutman index of $\Gamma\mathbb{Z}_{p^n}$ is*

$$Gut(\Gamma\mathbb{Z}_{p^n}) = \Big[(n+1)p^n - np^{n-1} - p^{\lceil\frac{n-1}{2}\rceil}\Big]^2 - \Big[(p^n-1)^2 + (p^n-p^{n-1})\left(\frac{(p^n-1)}{p-1} - 2\lceil\frac{n-1}{2}\rceil\right) + p^{\lceil\frac{n-1}{2}\rceil} - 1\Big] - \Big[(p^n-1)\left(n(p^n-p^{n-1}) - p^{\lceil\frac{n-1}{2}\rceil}+1\right) + \frac{1}{2}(p^{\lceil\frac{n-1}{2}\rceil)}-1)(p^{\lceil\frac{n-1}{2}\rceil}-2) + \frac{1}{2}(p^n-p^{n-1})\left(\frac{n(n-1)}{2}(p^n-p^{n-1}) + 2n + 2\lceil\frac{n-1}{2}\rceil - 1 - \frac{p^n-p+p^{\lceil\frac{n-1}{2}\rceil}-1}{p-1} - np^{\lceil\frac{n-1}{2}\rceil}\right)\Big].$$

Proof. Since we have determined $|E(\Gamma\mathbb{Z}_{p^n})|, M_1(\Gamma\mathbb{Z}_{p^n}), M_2(\Gamma\mathbb{Z}_{p^n})$ and by Lemma 2.6, Theorem 2.5, the result follows. □

2.3. Topological Indices of the Zero Divisor Graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$. .

This subsection contains the results for the Wiener index, the hyper-Wiener index, the Harary index, the edge-Wiener index, the first Zagreb index, the second Zagreb index and Gutman index for $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ for p,q prime numbers and $n,m\in\mathbb{N}$.

Proposition 2.23. *Let p,q be prime numbers, $n,m\in\mathbb{N}$ and $(a,b)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with $gcd(a,p^n)=p^i$ for $i=0,1,2,...,n$ and $gcd(b,q^m)=q^j$ for $j=0,1,2,...,m$. Then, the degree of (a,b) in the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ is*

$$deg((a,b)) = \begin{cases} p^iq^j, & i=0,1,2,...,n, j\le\lfloor\frac{m-1}{2}\rfloor \text{ or } i\le\lfloor\frac{n-1}{2}\rfloor, j>\lfloor\frac{m-1}{2}\rfloor \\ p^iq^j-1, & i>\lfloor\frac{n-1}{2}\rfloor, j>\lfloor\frac{m-1}{2}\rfloor \end{cases}$$

Proof. Let $(a,b)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with $\gcd(a,p^n)=p^i$, $\gcd(b,q^m)=q^j$ and $(c,d)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with $\gcd(c,p^n)=p^k$, $\gcd(d,q^m)=q^l$. Then, (a,b) is adjacent to (c,d) if and only if $k\ge n-i$ and $l\ge m-j$
If $i>\lfloor\frac{n-1}{2}\rfloor$, $j>\lfloor\frac{m-1}{2}\rfloor$ then $(a,b)\in p^{n-i}\mathbb{Z}_{p^n}\times q^{m-j}\mathbb{Z}_{q^m}$. So $deg((a,b)) = p^iq^j-1$.
If $0\le i\le n$, $j\le\lfloor\frac{m-1}{2}\rfloor$ or $i\le\lfloor\frac{n-1}{2}\rfloor$, $j\ge\lfloor\frac{m-1}{2}\rfloor$ then $deg((a,b))=p^iq^j$.
□

Proposition 2.24. *Let $V'_{ij}=\{(a,b)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m} : gcd(a,p^n)=p^i, gcd(b,q^m)=q^j\}$, then*

$$|V'_{ij}| = \begin{cases} (p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)}), & i=0,1,2,...,n-1, j=0,1,2,...,m-1 \\ (p^{n-i}-p^{n-(i+1)}), & i=0,1,2,...,n-1, j=m \\ (q^{m-j}-q^{m-(j+1)}), & i=n, j=0,1,2,...,m-1 \\ 1, & i=n.j=m \end{cases}$$

Proof. Since $|\{(a,b)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m} : gcd(a,p^n)=p^i, gcd(b,q^m)=q^j\}| = |\{a\in\mathbb{Z}_{p^n} : gcd(a,p^n)=p^i\}||\{b\in\mathbb{Z}_{q^m} : gcd(b,q^m)=q^j\}|$, the proof follows from Proposition 2.8. □

Lemma 2.25. *The number of edges of* $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ *is* $\frac{1}{2}\Big(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m+p^nq^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}\Big)$.

Proof. The number of edges in any graph will be equal to half of the sum of the degrees of all vertices in the graph. By using Proposition 2.23 and Proposition 2.24, we have,

$$\sum_{a\in V(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))} deg(a)$$
$$=\sum_{i=0}^{n-1}\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})p^iq^j+\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(q^{m-j}-q^{m-(j+1)})p^nq^j+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})p^iq^j+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})p^iq^m+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})(p^iq^j-1)+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^{n-i}-p^{n-(i+1)})(p^iq^m-1)+\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(q^{m-j}-q^{m-(j+1)})(p^nq^j-1)+p^nq^m-1$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(p^n-p^{n-1})(q^m-q^{m-1})+\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(q^m-q^{m-1})p^n+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^n-p^{n-1})(q^m-q^{m-1})+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(p^n-p^{n-1})q^m+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^n-p^{n-1})(q^m-q^{m-1})-\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^n-p^{n-1})q^m-\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^{n-i}-p^{n-(i+1)})+\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(q^m-q^{m-1})p^n-\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(q^{m-j}-q^{m-(j+1)})+p^nq^m-1$$
$$=nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}+p^nq^m.$$

Therefore, the number of edges is,

$$\frac{1}{2}\Big(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}+p^nq^m\Big).$$ □

Theorem 2.26. *Let* $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ *be the zero divisor graph of* $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ *with prime numbers* p,q *and natural numbers* n,m*. Then, the Wiener index of* $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ *is* $W(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))=p^nq^m(p^nq^m-1)-\frac{1}{2}(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m+p^nq^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil})$.

Proof. This is clear by using Lemma 2.1 and Theorem 2.25. □

Example 2.27. $W(\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_3))=2^2.3^1(2^2.3^1-1)-\frac{1}{2}\Big[2.1(2^2-2^{2-1})(3^1-3^{1-1})+1(3^1-3^{1-1})2^2+2(2^2-2^{2-1})3^1+2^2.3^1-2^{\lceil\frac{2-1}{2}\rceil}3^{\lceil\frac{1-1}{2}\rceil}\Big]=113.$

Theorem 2.28. *Let $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ be the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with prime numbers p,q and natural numbers n,m. Then, the hyper-Wiener index of $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ is* $WW(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))=\frac{3}{2}p^nq^m(p^nq^m-1)-(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m+p^nq^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil})$.

Proof. This is clear by using Theorem 2.2 and Lemma 2.25. □

Example 2.29. $WW(\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_3))=\frac{3}{2}.2^2.3^1(2^2.3^1-1)-\Big[2.1(2^2-2^{2-1})(3^1-3^{1-1})+1(3^1-3^{1-1})2^2+2(2^2-2^{2-1})3^1+2^2.3^1-2^{\lceil\frac{2-1}{2}\rceil}3^{\lceil\frac{1-1}{2}\rceil}\Big]=160.$

Theorem 2.30. *Let $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ be the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with prime numbers p,q and natural numbers n,m. Then, the Harary index of $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ is* $H(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))=\frac{1}{4}p^nq^m(p^nq^m-1)+\frac{1}{4}(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m+p^nq^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil})$.

Proof. This is clear by using Theorem 2.3 and Lemma 2.25. □

Example 2.31. $H(\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_3))=\frac{1}{4}.2^2.3^1(2^2.3^1-1)+\frac{1}{4}\Big[2.1(2^2-2^{2-1})(3^1-3^{1-1})+1(3^1-3^{1-1})2^2+2(2^2-2^{2-1})3^1+2^2.3^1-2^{\lceil\frac{2-1}{2}\rceil}3^{\lceil\frac{1-1}{2}\rceil}\Big]=\frac{85}{2}.$

Theorem 2.32. *Let $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ be the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with prime numbers p,q and natural numbers n,m. Then, the first Zagreb index of $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ is* $M_1(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))=(p^{2n-1}-p^{n-1})(q^{2m-1}-q^{m-1})+p^{2n}(q^{2m-1}-q^{m-1})+q^{2n}(p^{2n-1}-p^{n-1})-2\lceil\frac{n-1}{2}\rceil\lceil\frac{m-1}{2}\rceil(p^n-p^{n-1})(q^m-q^{m-1})+(p^{\lceil\frac{n-1}{2}\rceil}-1)(q^{\lceil\frac{m-1}{2}\rceil}-1)-2\lceil\frac{n-1}{2}\rceil(p^n-p^{n-1})q^m+(p^{\lceil\frac{n-1}{2}\rceil}-1)-2\lceil\frac{m-1}{2}\rceil(q^m-q^{m-1})p^n+(q^{\lceil\frac{m-1}{2}\rceil}-1)+(p^nq^m-1)^2$.

Proof. Using Proposition 2.23 and Proposition 2.24, we have,

$$M_1(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))=\sum_{a\in V(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))}deg(a)^2$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})(p^iq^j)^2+\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(q^{m-j}-q^{m-(j+1)})(p^nq^j)^2+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})(p^iq^j)^2+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(p^{n-i}-p^{n-(i+1)})(p^iq^m)^2+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(p^{n-i}-p^{n-(i+1)})(q^{m-j}-q^{m-(j+1)})(p^iq^j-1)^2+\sum_{i=1+\lfloor\frac{n-1}{2}\rfloor}^{n-1}(p^{n-i}-p^{n-(i+1)})(p^iq^m-1)^2+\sum_{j=1+\lfloor\frac{m-1}{2}\rfloor}^{m-1}(q^{m-j}-q^{m-(j+1)})(p^nq^j-1)^2+(p^nq^m-1)^2$$

$$=(p^n-p^{n-1})(q^m-q^{m-1})(\tfrac{p^n-1}{p-1})(\tfrac{q^m-1}{q-1})+(q^m-q^{m-1})p^{2n}(\tfrac{q^m-1}{q-1})+(p^n-p^{n-1})q^{2m}(\tfrac{p^n-1}{p-1})-2(n-1-\lfloor\tfrac{n-1}{2}\rfloor)(m-1-\lfloor\tfrac{m-1}{2}\rfloor)(p^n-p^{n-1})(q^m-q^{m-1})+(p^{n-(1+\lfloor\frac{n-1}{2}\rfloor)}-1)(q^{m-(1+\lfloor\frac{m-1}{2}\rfloor)}-1)-2(n-1-\lfloor\tfrac{n-1}{2}\rfloor)(p^n-p^{n-1})q^m+(p^{n-(1+\lfloor\frac{n-1}{2}\rfloor)}-1)-2(m-1-\lfloor\tfrac{m-1}{2}\rfloor)(q^m-q^{m-1})p^n+(q^{m-(1+\lfloor\frac{m-1}{2}\rfloor)}-1)+(p^nq^m-1)^2$$

$$=(p^{2n-1}-p^{n-1})(q^{2m-1}-q^{m-1})+p^{2n}(q^{2m-1}-q^{m-1})+q^{2n}(p^{2n-1}-p^{n-1})-2\lceil\tfrac{n-1}{2}\rceil\lceil\tfrac{m-1}{2}\rceil(p^n-p^{n-1})(q^m-q^{m-1})+(p^{\lceil\frac{n-1}{2}\rceil}-1)(q^{\lceil\frac{m-1}{2}\rceil}-1)-2\lceil\tfrac{n-1}{2}\rceil(p^n-p^{n-1})q^m+(p^{\lceil\frac{n-1}{2}\rceil}-1)-$$

$2\lceil \frac{m-1}{2}\rceil (q^m - q^{m-1})p^n + (q^{\lceil \frac{m-1}{2}\rceil} - 1) + (p^n q^m - 1)^2.$

□

Example 2.33. $M_1(\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_3)) = (2^{2.2-1}-2^{2-1})(3^{2.1-1}-3^{1-1})+2^{2.2}(3^{2.1-1}-3^{1-1}) + 3^{2.1}(2^{2.2-1}-2^{2-1}) - 2\lceil\frac{2-1}{2}\rceil\lceil\frac{1-1}{2}\rceil(2^2-2^{2-1})(3^1-3^{1-1}) + (2^{\lceil\frac{2-1}{2}\rceil}-1)(3^{\lceil\frac{1-1}{2}\rceil}-1) - 2\lceil\frac{2-1}{2}\rceil(2^2-2^{2-1})3^1 + (2^{\lceil\frac{2-1}{2}\rceil}-1) - 2\lceil\frac{1-1}{2}\rceil(3^1-3^{1-1})2^2 + (3^{\lceil\frac{1-1}{2}\rceil}-1)+(2^2.3^1-1)^2 = 6.2+16.2+9.6-0+0-2.1.2.3+1-0+0+121 = 208.$

Theorem 2.34. *Let $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ be the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ with prime numbers p, q and natural numbers n, m. Then, the edge-Wiener index of $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ is $W_e(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})) = |E(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))|^2 - \frac{1}{2}M_1(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))$.*

Proof. Let $(a_1,a_2),(b_1,b_2),(c_1,c_2),(d_1,d_2)\in\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$ be all different vertices in $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ such that $(a_1,a_2)(b_1,b_2) = (c_1,c_2)(d_1,d_2) = (0,0)$.
Then $((a_1,a_2),(b_1,b_2)),((c_1,c_2),(d_1,d_2)) \in E(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ and follows that $((a_1,a_2),(b_1,b_2)),((c_1,c_2),(d_1,d_2))$ are different vertices in $L(\Gamma\mathbb{Z}_{p^n})$.
Let $0\le i,j,k,l\le n$ and $0\le i',j',k',l'\le m$ be integers that satisfy $\gcd(a_1,p^n) = p^i$, $\gcd(a_2,q^m) = q^{i'}$, $\gcd(b_1,p^n) = p^j$, $\gcd(b_2,q^m) = q^{j'}$, $\gcd(c_1,p^n) = p^k$, $\gcd(c_2,q^m) = q^{k'}$, $\gcd(d_1,p^n) = p^l$, $\gcd(d_2,q^m) = q^{l'}$. Note that $i+j\ge n$, $i'+j'\ge m$ and $k+l\ge n$, $k'+l'\ge m$, so $\max\{i,j\}+\max\{k,l\}\ge n$ and $\max\{i',j'\}+\max\{k',l'\}\ge m$.
Choose $x_1\in\{a_1,b_1\}, x_2\in\{a_2,b_2\}$ and $y_1\in\{c_1,d_1\}, y_2\in\{c_2,d_2\}$ that satisfy $\gcd(x_1,p^n) = p^{\max\{i,j\}}$, $\gcd(x_2,q^m) = q^{\max\{i',j'\}}$ and $\gcd(y_1,p^n) = p^{\max\{k,l\}}$, $\gcd(y_2,q^m) = q^{\max\{k',l'\}}$. Hence $((x_1,x_2),(y_1,y_2)) \in E(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ and $((a_1,a_2),(b_1,b_2)) - ((x_1,x_2),(y_1,y_2)) - ((c_1,c_2),(d_1,d_2))$ is a path with length 2. So $\mathrm{diam}(L(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})))\le 2$.

By Theorem 2.4, we have,

$$W_e(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})) = |E(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}))|^2 - \frac{1}{2}M_1(\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})).$$

□

With the same technique as was used above for the first Zagreb index of $\Gamma\mathbb{Z}_{p^n}$, we get the first Zagreb index of $\Gamma(\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m})$ to be,
$(p^nq^m-1)\Big(nm(p^n-p^{n-1})(q^m-q^{m-1})+m(q^m-q^{m-1})p^n+n(p^n-p^{n-1})q^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}+1\Big) +\frac{1}{2}(m-1)(q^m-q^{m-1})p^n\Big(n(p^n-p^{n-1})q^m-p^{\lceil\frac{n-1}{2}\rceil}+1\Big)+\frac{1}{2}\frac{m(m-1)}{2}(q^m-q^{m-1})p^n\Big((n+1)p^n-np^{n-1})\Big)(q^m-q^{m-1}) -\frac{1}{2}\Big(\frac{q^{1+\lceil\frac{m-1}{2}\rceil}-q}{q-1}-\lceil\frac{m-1}{2}\rceil\Big)(q^m-q^{m-1})p^np^{\lceil\frac{n-1}{2}\rceil} -\frac{1}{2}(q^m-q^{m-1})p^n\Big(\lceil\frac{m-1}{2}\rceil p^{\lceil\frac{n-1}{2}\rceil}(q^{\lceil\frac{m-1}{2}\rceil}-1)-\lceil\frac{m-1}{2}\rceil+p^n\frac{q^{1+\lceil\frac{m-1}{2}\rceil}-q}{q-1}\Big) -\frac{1}{2}(q^{\lceil\frac{m-1}{2}\rceil}-1)\Big(n(p^n-p^{n-1})q^m-p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}+2\Big)+\frac{1}{2}\lceil\frac{m-1}{2}\rceil(q^m-q^{m-1})p^n+\frac{1}{2}\Big(m-(1+\lfloor\frac{m-1}{2}\rfloor)q^{\lceil\frac{m-1}{2}\rceil}-\frac{q^{\lceil\frac{m-1}{2}\rceil}-1}{q-1}\Big)\Big((n+1)p^n-np^{n-1}\Big)(q^m-q^{m-1}) +\frac{1}{2}(n-1)(p^n-p^{n-1})q^m\Big(m(q^m-q^{m-1})p^n-$

$$
\begin{aligned}
&q^{\lceil \frac{m-1}{2}\rceil}+1\bigg)+\frac{1}{2}\frac{n(n-1)}{2}(p^n-p^{n-1})q^m\bigg((m+1)q^m-mq^{m-1})\bigg)(p^n-p^{n-1})-\frac{1}{2}\bigg(\frac{p^{1+\lceil\frac{n-1}{2}\rceil}-p}{p-1}-\\
&\lceil\tfrac{n-1}{2}\rceil\bigg)(p^n-p^{n-1})q^m q^{\lceil\frac{m-1}{2}\rceil}-\frac{1}{2}(p^n-p^{n-1})q^m\bigg(\lceil\tfrac{n-1}{2}\rceil q^{\lceil\frac{m-1}{2}\rceil}(p^{\lceil\frac{n-1}{2}\rceil}-1)-\lceil\tfrac{n-1}{2}\rceil+\\
&q^m\frac{p^{1+\lceil\frac{n-1}{2}\rceil}-p}{p-1}\bigg)-\frac{1}{2}(p^{\lceil\frac{n-1}{2}\rceil}-1)\bigg(m(q^m-q^{m-1})p^n-q^{\lceil\frac{m-1}{2}\rceil}p^{\lceil\frac{n-1}{2}\rceil}+2\bigg)+\frac{1}{2}\lceil\tfrac{n-1}{2}\rceil(p^n-\\
&p^{n-1})q^m+\frac{1}{2}\bigg(n-(1+\lfloor\tfrac{n-1}{2}\rfloor)p^{\lceil\frac{n-1}{2}\rceil}-\frac{p^{\lceil\frac{n-1}{2}\rceil}-1}{p-1}\bigg)\bigg((m+1)q^m-mq^{m-1}\bigg)(p^n-p^{n-1})\\
&+\frac{1}{2}\frac{n(n-1)}{2}\frac{m(m-1)}{2}(p^n-p^{n-1})^2(q^m-q^{m-1})^2+\frac{1}{2}\frac{m(m-1)}{2}(p^n-p^{n-1})(q^m-q^{m-1})^2p^n+\\
&\frac{1}{2}\frac{n(n-1)}{2}(p^n-p^{n-1})^2(q^m-q^{m-1})q^m-\frac{1}{2}(p^n-p^{n-1})(q^m-q^{m-1})\bigg(\frac{p^{1+\lfloor\frac{n-1}{2}\rfloor}-p}{p-1}\frac{q^{1+\lfloor\frac{m-1}{2}\rfloor}-q}{q-1}-\\
&\lfloor\tfrac{n-1}{2}\rfloor\lfloor\tfrac{m-1}{2}\rfloor\bigg)-\frac{1}{2}(p^n-p^{n-1})(q^m-q^{m-1})\bigg(\frac{p^{1+\lfloor\frac{n-1}{2}\rfloor}-p}{p-1}q^{\lceil\frac{m-1}{2}\rceil}-\lfloor\tfrac{n-1}{2}\rfloor\lceil\tfrac{m-1}{2}\rceil+\frac{q^{1+\lfloor\frac{m-1}{2}\rfloor}-q}{q-1}\\
&p^{\lceil\frac{n-1}{2}\rceil}-\lceil\tfrac{n-1}{2}\rceil\lfloor\tfrac{m-1}{2}\rfloor+\lceil\tfrac{n-1}{2}\rceil\lceil\tfrac{m-1}{2}\rceil(p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}-2)+\frac{p^{1+\lceil\frac{n-1}{2}\rceil}-p}{p-1}\frac{q^{1+\lceil\frac{m-1}{2}\rceil}-q}{q-1}\bigg)\\
&+\frac{1}{2}\lceil\tfrac{n-1}{2}\rceil\lceil\tfrac{m-1}{2}\rceil(p^n-p^{n-1})(q^m-q^{m-1})+\frac{1}{2}(p^{\lceil\frac{n-1}{2}\rceil}-1)(q^{\lceil\frac{m-1}{2}\rceil}-1)(p^{\lceil\frac{n-1}{2}\rceil}q^{\lceil\frac{m-1}{2}\rceil}-2)\\
&-\frac{1}{2}\bigg(n-(1+\lfloor\tfrac{n-1}{2}\rfloor)p^{\lceil\frac{n-1}{2}\rceil}-\frac{p^{\lceil\frac{n-1}{2}\rceil}-1}{p-1}\bigg)\bigg(m-(1+\lfloor\tfrac{m-1}{2}\rfloor)q^{\lceil\frac{m-1}{2}\rceil}-\frac{q^{\lceil\frac{m-1}{2}\rceil}-1}{q-1}\bigg)(p^n-\\
&p^{n-1})(q^m-q^{m-1})-\frac{1}{2}\bigg(n-(1+\lfloor\tfrac{n-1}{2}\rfloor)p^{\lceil\frac{n-1}{2}\rceil}-\frac{p^{\lceil\frac{n-1}{2}\rceil}-1}{p-1}\bigg)(q^{\lceil\frac{m-1}{2}\rceil}-1)(p^n-p^{n-1})q^m-\\
&\frac{1}{2}\bigg(m-(1+\lfloor\tfrac{m-1}{2}\rfloor)q^{\lceil\frac{m-1}{2}\rceil}-\frac{q^{\lceil\frac{m-1}{2}\rceil}-1}{q-1}\bigg)(p^{\lceil\frac{n-1}{2}\rceil}-1)(q^m-q^{m-1})p^n+\frac{1}{2}\frac{m(m-1)}{2}(p^n-\\
&p^{n-1})(q^m-q^{m-1})(q^m-q^{m-1})p^n-\frac{1}{2}(p^n-p^{n-1})(q^m-q^{m-1})(\frac{q^{1+\lfloor\frac{m-1}{2}\rfloor}-q}{q-1}-\lfloor\tfrac{m-1}{2}\rfloor)-\\
&\frac{1}{2}\lceil\tfrac{m-1}{2}\rceil(p^n-p^{n-1})(q^m-q^{m-1})(q^{\lceil\frac{m-1}{2}\rceil}-1)+\frac{1}{2}(p^n-p^{n-1})q^m\bigg(m(q^m-q^{m-1})p^n-(q^{\lceil\frac{m-1}{2}\rceil}-\\
&1)\bigg)+\frac{1}{2}\frac{n(n-1)}{2}(q^m-q^{m-1})(p^n-p^{n-1})(p^n-p^{n-1})q^m-\frac{1}{2}(q^m-q^{m-1})(p^n-p^{n-1})(\frac{p^{1+\lfloor\frac{n-1}{2}\rfloor}-p}{p-1}-\\
&\lfloor\tfrac{n-1}{2}\rfloor)-\frac{1}{2}\lceil\tfrac{n-1}{2}\rceil(q^m-q^{m-1})(p^n-p^{n-1})(p^{\lceil\frac{n-1}{2}\rceil}-1)+\frac{1}{2}(q^m-q^{m-1})p^n\bigg(n(p^n-p^{n-1})q^m-\\
&(p^{\lceil\frac{n-1}{2}\rceil}-1)\bigg).
\end{aligned}
$$

3. Conclusion

In this paper, we have determined the Wiener index, the edge Wiener index, the hyper Wiener index, the Harary index, the first Zagreb index, the second Zagreb index and the Gutman index of the zero divisor graph of $\mathbb{Z}_{p^n}$ and the zero divisor graph of $\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^m}$. In future research, we will determine the topological indices of the zero divisor graph of $\mathbb{Z}_{p_1^{k_1}}\times\mathbb{Z}_{p_2^{k_2}}\times ...\times\mathbb{Z}_{p_m^{k_m}}$ for $m>2$.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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