On Some Properties of Algebraic Graphs Associated to Groups and Rings

PROF. DR. NOR HANIZA SARMIN
Department of Mathematical Sciences, Faculty of Science,
Universiti Teknologi Malaysia, Johor Bahru, Johor Malaysia

INSPEM'S MONTHLY SEMINAR (05/2024)

24 July 2024

UPM UNIVERSITI PUTRA MALAYSIA BERILMU BERBAKTI

INSTITUT PENYELIDIKAN MATEMATIK
INSTITUTE FOR MATHEMATICAL RESEARCH
اينستيتوت ڤڽليديقن ماتماتيك

MONTHLY SEMINAR

24 JULY 2024 | 2.30PM-4.30PM

HYBRID: FACE TO FACE AND GOOGLE MEET

LINK GOOGLE MEET: HTTPS://MEET.GOOGLE.COM/DMV-VJWN-NCI

MEET OUR SPEAKER

PROF. DR. SHAKIR ALI
DEPARTMENTS OF MATHEMATICS ALIGARH MUSLIM UNIVERSITY, INDIA

TOPICS: ON SYMMETRIC DERIVATIONS IN RINGS

YBHG. PROF. DR. NOR HANIZA BINTI SARMIN
UNIVERSITI TEKNOLOGI MALAYSIA (UTM), JOHOR BHARU, JOHOR
FACULTY SCIENCES

TOPICS: ON SOME PROPERTIES OF ALGEBRAIC GRAPHS ASSOCIATED TO GROUPS AND RINGS

ABSTRACT

Let R be any ring. An additive mapping d : R → R is said to be a derivation on R if d(xy) = d(x)y + xd(y) holds for all x, y ∈ R. A bi-additive map D : R × R → R is said to be a bi-derivation if D(xx', y) = D(x, y)x' + xD(x' , y) D(x, yy') = D(x, y)y' + yD(x, y') holds for any x, x', y, y' ∈ R. The foregoing conditions are identical if D is also a symmetric map, that is, if D(x, y) = D(y, x) for every x, y ∈ R. In this case, D referred as a symmetric bi-derivation on R. In this talk, we will discuss the recent progress made on the topic and related areas. Moreover, some examples and counter examples will be discussed for questions raised naturally. We conclude the talk with some open problems.

ABSTRACT

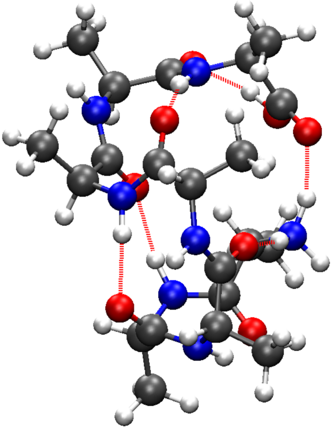
In this talk, some properties of algebraic graphs associated to some groups and rings, namely the energies of the graphs and their topological indices, will be presented. The energy of a simple graph is defined as the summation of the absolute value of the eigenvalues of the adjacency matrix of the graph. It was motivated by the Hckel Molecular Orbital theory. The theory was used by chemists to estimate the energy associated with -electron orbitals of molecules which is called conjugated hydrocarbons. Meanwhile, a topological index is a function that assigns a numeric value to a (molecular) graph that predicts its various physical and structural properties such as volume, density, pressure, weight, boiling point, freezing point, vaporisation point, heat of formation, and heat of evaporation. In this presentation, the energy and Laplacian energy of the non-commuting and conjugacy class graphs associated to some finite groups are presented. The Seidel energy of the Cayley graph of some finite groups are also determined. In addition, this presentation focuses on the degree-based and distance-based topological indices. The degree-based topological indices include the first Zagreb index, the second Zagreb index, the general zeroth-order Randi index, and the Sombor index. The distance-based topological indices include the Wiener index, the Szeged index, and the Harary index. The graphs considered are the non-commuting graph and the coprime graph associated to the dihedral groups, the generalized quaternion groups, the quasidihedral groups, alternating groups, and symmetric groups. Another graph discussed in finding its topological indices is the zero divisor graph associated to some commutative rings.

Topological Indices of Algebraic Graphs Associated to Groups and Rings

Abstract

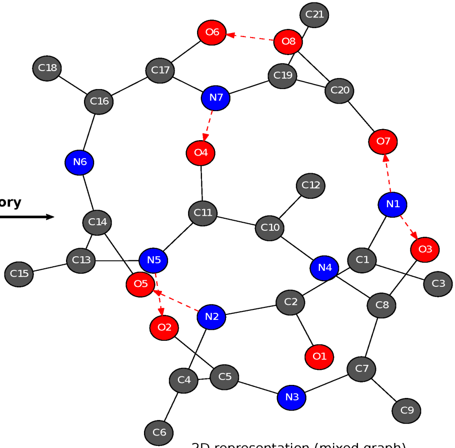
In this talk, some properties of algebraic graphs associated to some groups and rings, namely the energies of the graphs and their topological indices, will be presented. The energy of a simple graph is defined as the summation of the absolute value of the eigenvalues of the adjacency matrix of the graph. It was motivated by the Hückel Molecular Orbital theory. The theory was used by chemists to estimate the energy associated with π-electron orbitals of molecules which is called conjugated hydrocarbons. Meanwhile, a topological index is a function that assigns a numeric value to a (molecular) graph that predicts its various physical and structural properties such as volume, density, pressure, weight, boiling point, freezing point, vaporisation point, heat of formation, and heat of evaporation. In this presentation, the energy and Laplacian energy of the non-commuting and conjugacy class graphs associated to some finite groups are presented. The Seidel energy of the Cayley graph of some finite groups are also determined. In addition, this presentation focuses on the degree-based and distance-based topological indices. The degree-based topological indices include the first Zagreb index, the second Zagreb index, the general zeroth-order Randić index, and the Sombor index. The distance-based topological indices include the Wiener index, the Szeged index, and the Harary index. The graphs considered are the non-commuting graph and the coprime graph associated to the dihedral groups, the generalized quaternion groups, the quasidihedral groups, alternating groups, and symmetric groups. Another graph discussed in finding its topological indices is the zero divisor graph associated to some commutative rings.

Motivation of the Research



3D representation (position)

Graph theory →



2D representation (mixed graph)

Presentation Outline

Introduction to the Energy and Laplacian Energy of a Graph

- The energy of graph was first defined by Gutman in 1978 as the sum of the absolute values of the eigenvalues of the graph [1].
- The motivation for his definition comes from chemistry. It is used to approximate the total π-electron energy of molecules.
- On the other hand, the Laplacian energy of the graph was first defined by Gutman and Zhou in 2006 [2].
- Recently, the energy of a graph has become a quantity of interest to mathematicians, where several variations have been introduced.
- Our focus: The energy and the Laplacian energy of some graphs related to some finite groups.

[1] I. Gutman, *The energy of a graph, Der. Math. stat. Sekt. Forschungszent*, (1978) 1-22.
[2] I. Gutman and B. Zhou, *Laplacian energy of a graph, Linear Algebra and its Applications*, **414(1)** (2006), 29-37.

Some Definitions and Concepts in Graph Theory

Adjacency Matrix of a Graph [3]

Let Γ be a graph with the vertex-set $V(\Gamma) = \{1, \ldots, n\}$; and the edge set $E(\Gamma) = \{e_1, \ldots, e_m\}$. The adjacency matrix of Γ, denoted by $A(\Gamma)$, is an $n \times n$ matrix defined as follows. The rows and the columns of $A(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$, then the (i, j)-entry of $A(\Gamma)$ is 0 for non adjacent vertices i and j, and the (i, j)-entry is 1 for adjacent i and j. The (i, i)-entry of $A(\Gamma)$ is 0 for $i = 1, \ldots, n$.

Energy of Graph [1]

For any graph Γ, the energy of the graph is defined as $\varepsilon(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of Γ.

[1] I. Gutman, *The energy of a graph, Der. Math. stat. Sekt. Forschungszent*, (1978) 1-22.
[3] R.B. Bapat, *Graphs and matrices, Springer*, **27** (2010).

Some Definitions and Concepts in Graph Theory

Laplacian Matrix of a Graph [3]

Let Γ be a graph with the vertex-set $V(\Gamma) = \{1, \ldots, n\}$ and the edge-set, $E(\Gamma) = \{e_1, \ldots, e_m\}$. The Laplacian matrix of Γ, denoted by $L(\Gamma)$, is an $n \times n$ matrix defined as follows: the rows and the columns of $L(\Gamma)$ are indexed by $V(\Gamma)$. If $i \neq j$, then a_{ij} is 0 if vertex *i* and *j* are not adjacent, and it is -1 if *i* and *j* are adjacent. The a_{ii} entry of $L(\Gamma)$ is d_i, the degree of vertex *i*, *i*=1,2, 3,. . . ,*n*.

Laplacian Energy of Graph [2]

Let Γ be a simple graph, L be its Laplacian matrix and μ_1,μ_2, . . . ,μ_n be the eigenvalues of the Laplacian matrix. Then, the Laplacian energy of the graph Γ, denoted by $LE(\Gamma)$, is $LE(\Gamma) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where n is the number of vertices of the graph $\Gamma$and m is the number of its edges.

[2] I. Gutman and B. Zhou, *Laplacian energy of a graph, Linear Algebra and its Applications*, **414(1)** (2006), 29-37.
[3] R.B. Bapat, *Graphs and matrices, Springer*, **27** (2010).

Some Definitions and Concepts in Graph Theory

Seidel Matrix [4]

The Seidel matrix of a simple graph Γ with n vertices and m edges, denoted by $S(G) = (s_{ij})$ is a real square symmetric matrix of order n defined as $s_{ij} = -1$ if v_i and v_j are adjacent, $s_{ij} = 1$ if v_i and v_j are not adjacent and 0 if $i = j$.

Seidel Energy of Graph [5]

For any graph Γ, the Seidel energy of the graph is defined as $SE(\Gamma) = \sum_{i=1}^{n} |\theta_i|$ where $\theta_1, \ldots, \theta_n$ are the eigenvalues of the Seidel matrix $S(\Gamma)$ of Γ.

Non-commuting Graph [6]

Let G be a finite group. The non-commuting graph of G, denoted by Γ_G, is the graph of vertex set $G - Z(G)$ and two distinct vertices x and y are joined by an edge whenever $xy \neq yx$.

[4] J. P. Liu, , and B. L. Liu, *Generalization for Laplacian energy, Applied Mathematics-A Journal of Chinese Universities*, **24(4)** (2009), 443.
[5] W. H. Haemers, *Seidel switching and graph energy*, Available at SSRN 2026916, (2012).
[6] A. Abdollahi, S. Akbari, H. Maimani, *Non-commuting graph of a group. Journal of Algebra.* **298** (2006), 468-492.

Some Definitions and Concepts in Graph Theory

Conjugacy Class Graph [7]

Let G be a finite group. A conjugacy class graph of G, denoted as Γ_G^{cl}, is a graph with the vertices $V = (v_1, v_2, \ldots, v_n)$ represented by the non-central conjugacy classes of G. Two vertices v_i and v_j are adjacent if and only if $|v_i|$ and $|v_j|$ have a common prime divisor.

Cayley Graph of a Group [8]

Let G be a finite group with identity 1. Let X be a subset of G satisfying $1 \notin X$ and $X = X^{-1}$; that is, $x \in X$ if and only if $x^{-1} \in X$. The Cayley graph $Cay(G, X)$ on G with connection set X is defined as follows:

- the vertices are the elements of G
- there is an edge joining g_1 and g_2 if and only if $g_2 = xg_1$ for some $x \in X$.

The set of all Cayley graphs on G is denoted by $Cay(G, X)$.
Remark: The relation between the two vertices can be rewritten as $g_2 g_1^{-1} = x$ for some $x \in X$.

[7] E. A. Bertram, M. Herzog, and A. Mann, *On a graph related to conjugacy classes of groups, Bulletin of the London Mathematical Society*, **22**(6) (1990), 569–575.
[8] L. W. Beineke, and R. J. Wilson, *Topics in algebraic graph theory*, **102**,USA : Cambridge University Press, (2004).

Definition for Some Finite Groups

Dihedral Group [9]

The dihedral group, denoted by D_{2n}, is a group of symmetries of a regular polygon, which include rotations and reflections. The order of D_{2n} is $2n$, where $n \geq 3$ is an integer. The dihedral group can be presented in the form of generators and relations given as follows:

$$D_{2n} = \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle,$$

where a and b are the generators of D_{2n}.

Generalized Quaternion Group [9]

The generalized quaternion group, Q_{4n} is a group of order $4n$, $n \in \mathbb{N}$ and the group presentation of Q_{4n} is presented as follows:
$Q_{4n} = \langle a, b \mid a^n = b^2, a^{2n} = b^4 = 1, bab = a^{-1} \rangle$, where $n \geq 2$.

[9] J. Humphreys, *A course in group theory, vol. 6, Oxford University Press on Demand*, 1996.

Definition for Some Finite Groups

Quasidihedral Group [9]

The Quasidihedral Group, QD_{2^n}, is a non-abelian group of order 2^n with group presentation given as follows::

$$QD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \rangle,$$

where $n \in \mathbb{N}$ and $n \geq 4$.

[9] J. Humphreys, *A course in group theory, vol. 6, Oxford University Press on Demand,* 1996.

THE ENERGY AND SEIDEL ENERGY OF GRAPHS ASSOCIATED TO GROUPS

The Energy of Conjugacy Class Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 1 [10]

Let $G = D_{2n}$ be a dihedral group of order 2*n*, where $n \geq 3$, $n \in \mathbb{Z}^+$, i.e. $D_{2n} = \left\langle a, b | a^n = b^2 = 1, bab = a^{-1} \right\rangle$. Then, the conjugacy class graph of D_{2n} is as follows:

$$\Gamma_G^{cl} = \begin{cases} K_{\frac{n-1}{2}} \bigcup cl(b), & \text{if } n = \text{is odd,} \\ K_{\frac{n+2}{2}}, & \text{if } n \text{ and } \frac{n}{2} \text{ are even,} \\ K_{\frac{n-2}{2}} \bigcup K_2, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Theorem 2 [10]

Let G be the generalised quaternion groups, Q_{4n} of order $4n$ where $n \geq 2, n \in \mathbb{N}$. Then, the conjugacy class graph of G is

$$\Gamma_G^{cl} = \begin{cases} K_{n-1} \bigcup K_2, & \text{if} \quad n \, \text{is odd,} \\ K_{n+1}, & \text{if} \quad n \, \text{is even} \, . \end{cases}$$

The Energy of Conjugacy Class Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 3 [10]

Let G be the quasidihedral groups, QD_{2^n} of order 2^n where $n \geq 4, n \in \mathbb{N}$. Then, the conjugacy class graph of G is $\Gamma_G^{cl} = K_{2^{n-2}+1}$ where it is a complete graph with $2^{n-2} + 1$ vertices.

Theorem 4 [10]

Let G be a dihedral group of order $2n$. where n is an odd integer and $n \geq 3$, and let $\Gamma_{D_{2n}}^{cl}$ be its conjugacy class graph. Then, the energy of the graph $\Gamma_{D_{2n}}^{cl}$, $\varepsilon(\Gamma_{D_{2n}}^{cl}) = n - 3$.

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Conjugacy Class Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 5 [10]

Let G be a dihedral group of order $2n$ i.e. $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$, where n and $\frac{n}{2}$ are even integers and let $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph. Hence, the energy of the graph $\Gamma^{cl}_{D_{2n}}$,

$$\varepsilon \left(\Gamma^{cl}_{D_{2n}} \right) = n.$$

Theorem 6 [10]

Let G be a dihedral group of order $2n$, i.e. $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$, where n is an even integer and $\frac{n}{2}$ is an odd integer, and let $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph. Then, the energy of the graph $\Gamma^{cl}_{D_{2n}}$,

$$\varepsilon \left(\Gamma^{cl}_{D_{2n}} \right) = n - 2.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 7 [10]

Let G be the dihedral groups of order $2n$ where $n \geq 3, n \in \mathbb{N}$ and let Γ_G^{NC} be the non-commuting graph of G. Then,

$$\Gamma_G^{NC} = \begin{cases} K_{\underbrace{1,1,\ldots,1}_{\text{n times}},n-1}, & \text{if} \quad n \text{ is odd,} \\ K_{\underbrace{2,2,\ldots,2}_{\frac{n}{2} \text{ times}},n-2}, & \text{if} \quad n \text{ is even.} \end{cases}$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 8 [10]

Let G be the generalized quaternion groups of order $4n$ where $n \geq 2, n \in \mathbb{N}$ and let Γ_G^{NC} be the non-commuting graph of G. Then,

$$\Gamma_G^{\mathrm{NC}} = K_{\underbrace{2,\,2,\,\ldots,\,2}_{\text{n times}},2n-2}.$$

Theorem 9 [10]

Let G be the quasidihedral groups of order 2^n where $n \geq 4, n \in \mathbb{N}$ and let Γ_G^{NC} be the non-commuting graph of G. Then,

$$\Gamma_G^{\mathrm{NC}} = K_{\underbrace{2,\,2,\,\ldots,\,2}_{2^{n-2}\text{times}},2^{n-1}-2}.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 10 [10]

Let G be a dihedral group of order $2n$, where n is an odd integer, i.e. $G = D_{2n} \cong \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle$ and let $\Gamma_{D_{2n}}$ be its non-commuting graph. Then, the energy of the graph $\Gamma_{D_{2n}}^{NC}$ is

$$\varepsilon\left(\Gamma_{D_{2n}}^{NC}\right) = (n-1) + \sqrt{5n^2 - 6n + 1}$$

Theorem 11 [10]

Let G be a dihedral group of order $2n$ where n is an even integer , i.e. $G = D_{2n} \cong \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle$ and let $\Gamma_{D_{2n}}^{ncom}$ be its non-commuting graph. Then, the energy of the graph $\Gamma_{D_{2n}}^{NC}$ is

$$\varepsilon\left(\Gamma_{D_{2n}}^{NC}\right) = (n-2) + \sqrt{5n^2 - 12n + 4}.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 12 [10]

Let $G = Q_{4n}$ be a generalized quaternion group of order $4n$, i.e. $Q_{4n} = \left\langle a, b |\, a^n = b^2,\, a^{2n} = 1 = b^4,\, b^{-1}ab = a^{-1} \right\rangle$, where $n \geq 2$, and let $\Gamma^{NC}_{Q_{4n}}$ be its non-commuting graph. Then, the energy of $\Gamma^{NC}_{Q_{4n}}$ is

$$\varepsilon\left(\Gamma^{NC}_{Q_{4n}}\right) = (2n-2) + 2\sqrt{5n^2 - 6n + 1}.$$

Theorem 13 [10]

Let $G = QD_{2^n}$ be a quasidihedral group of order 2^n i.e. $QD_{2^n} = \langle a, b |$ $a^{2^{n-1}} = b^2 = 1 =, bab^{-1} = a^{2^{n-2}-1} \rangle$, where $n \geq 4$. Then, the energy of the non-commuting graph of $G = QD_{2^n}$ is

$$\varepsilon(\Gamma^{NC}_{QD_{2^n}}) = (2^{n-1} - 2) + 2\sqrt{(2^{n-2} - 1)(2^n + 2^{n-2} - 1)}.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Laplacian Spectrum of the Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 14 [10]

Consider that D_{2n} is a dihedral group of order $2n$ i.e. $D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$, where n is an even integer and $n > 4$. Then, the Laplacian spectrum of the non-commuting graph of D_{2n} is given as

$$L\text{-}spect\left(\Gamma^{NC}_{D_{2n}}\right) = \left\{(0)^1, (n)^{n-3}, (2n-2)^{\frac{n}{2}}, (2n-4)^{\frac{n}{2}}\right\}.$$

Theorem 15 [10]

Consider that Q_{4n} is a generalized quaternion group of order $4n$, where $n \geq 2$. Then, the Laplacian spectrum of the non-commuting graph of Q_{4n} is

$$L\text{-}Spect\left(\Gamma^{NC}_{Q_{4n}}\right) = \left\{(0)^1, (2n)^{2n-3}, (4n-2)^n, (4n-4)^n\right\}.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Laplacian Spectrum of the Non-commuting Graphs of D_{2n},Q_{4n} and QD_{2^n}

Theorem 16 [10]

Consider that $G = QD_{2^n}$ is a quasidihedral group of order 2^n, where $n \geq 4$. Then, the Laplacian spectrum of the non-commuting graph of QD_{2^n} is

$$L\text{-}Spect\left(\Gamma_{QD_{2^n}}^{NC}\right) = \left\{(0)^1, \left(2^{n-1}\right)^{2^{n-1}-3}, (2^n-2)^{2^{n-2}}, (2^n-4)^{2^{n-2}}\right\}.$$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Laplacian Energy of the Conjugacy Class Graphs of D_{2n},Q_{4n} and QD_{2^n}

Group		Laplacian Energy of the Conjugacy Class Graph of a Group, $LE(\Gamma_G^{cl})$
Dihedral Group, D_{2n}	n odd, $n \geq 3$	$LE(\Gamma_{D_{2n}}^{cl}) = n - 3$
	n and $\frac{n}{2}$ are even	$LE(\Gamma_{D_{2n}}^{cl}) = n$
	n even and $\frac{n}{2}$ odd	$LE(\Gamma_{D_{2n}}^{cl}) = n - 2$
Generalized Quaternion Group, Q_{4n}	n even, $n \geq 2$	$LE(\Gamma_{Q_{4n}}^{cl}) = 2n$
	n odd, $n \geq 2$	$LE(\Gamma_{Q_{4n}}^{cl}) = 2n - 2$
Quasihedral Group, QD_{2^n}	$n \geq 4$	$LE(\Gamma_{QD_{2^n}}^{cl}) = 2^{n-1}$

[10] R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

The Energy of Cayley Graphs Associated to Dihedral Groups

Theorem 17 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ be a subset of order one of D_{2n}. Then, the energy of the Cayley graphs $Cay(D_{2n}, X^{(1)})$ is $2n$.

Theorem 18 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ be a subset of order two of D_{2n}. Then, the energy of the Cayley graphs $Cay(D_{2n}, \{a, a^{n-1}\})$ is $\sum_{i=0}^{n-1} |4\cos(\frac{2i\pi}{n})|$.

Theorem 20 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 5$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $\sum_{i=0}^{n-1} |4\cos(\frac{2i\pi}{n})|$.

[11] A. F. Ahmad Fadzil, *The Energy and Seidel Energy of Cayley Graphs Associated to Dihedral, Alternating and Symmetric Groups*, Ph.D Thesis, Universiti Teknologi Malaysia, 2021.

The Energy of Cayley Graphs Associated to Dihedral Groups

Theorem 21 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 6$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $\sum_{i=0}^{n-1} |8\cos(\frac{4i\pi}{n})|$.

Theorem 22 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 4$ and n is even. Let $X^{(2)} = \{a^{\frac{n}{2}}, a^i b\}$ be a subset of order two of D_{2n}. Then, the energy of the Cayley graphs $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ is $\sum_{j=0}^{3} |n\cos(\frac{\pi j}{2})|$.

Theorem 23 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating set of D_{2n}. The energy of the Cayley graphs of D_{2n} with respect to the generating set X, $\varepsilon(Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})) = 2n$.

The Seidel Energy of the Cayley Graphs Associated to Dihedral Groups

Theorem 24 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(1)}$ be a subset of order one of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, X^{(1)})$ is $2n$.

Theorem 25 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X^{(2)} = \{a, a^{n-1}\}$ be a subset of order two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a, a^{n-1}\})$ is $\sum_{i=1}^{n} |8\cos(\frac{2\pi i}{n})|$.

Theorem 26 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 5$ and n is odd. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ is $\sum_{i=1}^{n} |8\cos(\frac{2\pi i}{n})|$.

The Seidel Energy of the Cayley Graphs Associated to Dihedral Groups

Theorem 27 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 6$ and n is even. Let $X^{(2)} = \{a^2, a^{n-2}\}$ be a subset of order two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^2, a^{n-2}\})$ for n even is $\sum_{i=1}^{n} |16\cos(\frac{4i\pi}{n})|$.

Theorem 28 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 4$ and n is even. Let $X^{(2)} = \{a^{\frac{n}{2}}, a^i b\}$ be a subset of order two of D_{2n}. Then, the Seidel energy of the Cayley graphs $Cay(D_{2n}, \{a^{\frac{n}{2}}, a^i b\})$ is $\sum_{j=0}^{3} |2n\cos(\frac{\pi j}{2})|$.

Theorem 29 [11]

Let D_{2n} be the dihedral group of order $2n$, where $n \geq 3$ and $X = \{b, ab, \ldots, a^{n-1}b\}$ be the generating set of D_{2n}. The Seidel energy of the Cayley graphs of D_{2n} with respect to the generating set X, $SE(Cay(D_{2n}, \{b, ab, \ldots, a^{n-1}b\})) = 4n - 2$.

Introduction to Topological Indices

- Topological indices provide numerical descriptors that capture important structural features of molecules.
- They serve as powerful tools for the analysis and prediction of various physicochemical properties and biological activities.
- The significance of topological indices lies in their ability to transform complex molecular structures into numerical representations,enabling the development of computational models and the efficient exploration of chemical space for various applications in drug discovery, materials science, and reaction chemistry [12].
- Various types of topological indices have been developed based on either chemistry or mathematical perspectives.

[12] I. Gutman and O.E. Polansky, *Mathematical Concepts in Organic Chemistry, Springer Science and Business Media*, 2012.

Types of Topological Indices

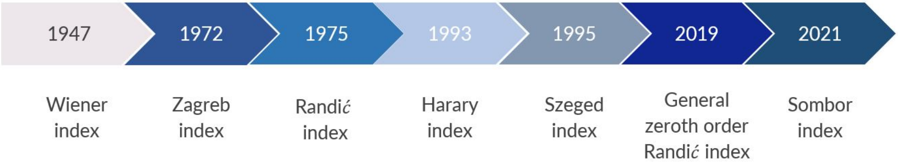


Figure 1: Different Types of Topological Indices

Definitions of Topological Indices

Wiener Index

- The first type of topological index has been discovered by Wiener [13] in 1947, in which the concept of Wiener number considering the path in a graph is introduced.
- The Wiener number of some paraffins are determined and their boiling points are also predicted.
- Then, Hosoya [14] reformulated the formula of Wiener number, known as Wiener index of a graph, $W(\Gamma)$, and its formula is given in the following.

$$W(\Gamma) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} d(i,j),$$

where $d(i,j)$ is the distance between vertices i and j, and m is the total number of vertices in a graph Γ.

[13] K. Wiener, *Structural determination of paraffin boiling points, Journal of the American Chemical Society*, **69(1)** (1947), 17-20.
[14] H. Hosoya, *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bulletin of the Chemical Society of Japan*, **44(9)** (1971), 2332–2339.

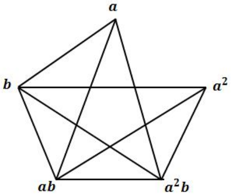


Figure 2: The non-commuting graph D_6

$$
\begin{aligned}
W(\Gamma) &= \frac{1}{2}\sum_{i=1}^{5}\sum_{j=1}^{5} d(i,j) \\
&= \frac{1}{2}\left[d(a,a^2) + d(a,b) + d(a,ab) + d(a,a^2b) + d(a^2,b) + \right. \\
&\quad \left. d(a^2,ab) + d(a^2,a^2b) + d(b,ab) + d(ab,a^2b) + d(b,a^2b)\right] \\
&= 2 + 9(1) \\
&= 11.
\end{aligned}
$$

Wiener Index of the Non-commuting Graph

Proposition 1 [15]

Let G be a finite group and Γ_G^{NC} be the non-commuting graph. Then, the Wiener index of the non-commuting graph of G is given as

$$W(\Gamma_G^{\mathrm{NC}}) = \frac{1}{2}\left[(|G| - |Z(G)|)\left(|G| - 2|Z(G)| - 2\right) + |G|\left(k(G) - |Z(G)|\right)\right].$$

[15] A. Azad and M. Eliasi, *Distance in the non-commuting graph of groups. Ars Comb.* **99** (2011), 279-287.

Zagreb Index

In 1972, Gutman and Trinajstić [16] introduced the degree-based topological index, Zagreb index, which is divided into two types; first Zagreb index, M_1, and second Zagreb index, M_2, defined as follows.

$$M_1(\Gamma) = \sum_{v \in v(\Gamma)} (\deg(v))^2$$

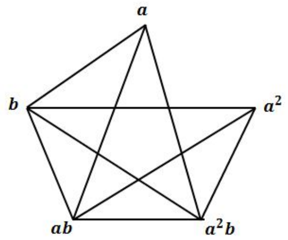
and

$$M_2(\Gamma) = \sum_{\{u,v\} \in E(\Gamma)} \deg(u)\deg(v).$$

[16] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, *Chemical Physics Letters*, **17(4)** (1972), 535-538.

Example of First Zagreb Index

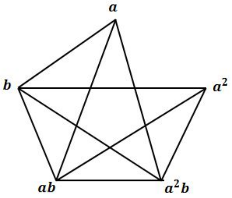
Based on Figure 2,



$$
\begin{aligned}
M_1(\Gamma) &= \sum_{v \in v(\Gamma)} (\deg(v))^2 \\
&= deg(a)^2 + deg(a^2)^2 + deg(b)^2 + deg(ab)^2 + deg(a^b)^2 \\
&= 3^2 + 3^2 + 4^2 + 4^2 + 4^2 \\
&= 66.
\end{aligned}
$$

Example of Second Zagreb Index

Based on Figure 2,



$$
\begin{aligned}
M_2(\Gamma) &= \sum_{\{u,v\} \in E(\Gamma)} \deg(u)\deg(v) \\
&= deg(a)deg(b) + deg(a)deg(ab) + deg(a)deg(a^2b) + deg(a^2)deg(b) + \\
&\quad deg(a^2)deg(ab) + deg(a^2)deg(a^2b) + deg(b)deg(ab) + deg(ab)deg(a^2b) \\
&\quad + deg(b)deg(a^2b) \\
&= 3(4) + 3(4) + 3(4) + 3(4) + 3(4) + 3(4) + 4(4) + 4(4) + 4(4) \\
&= 120.
\end{aligned}
$$

Zagreb Index of the Non-commuting Graph

Proposition 2 [17]

Let G be a finite group and Γ_G^{NC} be the non-commuting graph of G. Then, the first Zagreb index of the non-commuting graph of G,

$$M_1(\Gamma_G^{\mathrm{NC}}) = |G|^2(|G| + |Z(G)| - 2k(G)) - \sum_{x \in G - Z(G)} |C_G(x)|^2.$$

Proposition 3 [17]

Let G be a finite group and Γ_G^{NC} be the non-commuting graph. Then, the second Zagreb index of the non-commuting graph of G,

$$M_2(\Gamma_G^{\mathrm{NC}}) = -|G|^2|E(\Gamma_G^{\mathrm{NC}})| + |G|M_1(\Gamma_G^{\mathrm{NC}}) + \sum_{x,y \in E(\Gamma_G^{\mathrm{NC}})} |C_G(x)||C_G(y)|.$$

[17] M. Mizargar and A. Ashrafi, *Some distance-based topological indices of a non-commuting graph. Hacettepe Journal of Mathematics ans Staristics.* **41(4)** (2012), 515-526.

Definitions of Topological Indices

Szeged Index

Let Γ be a simple connected graph with vertex set $V(\Gamma) = \{1, 2, \ldots, n\}$. The Szeged index, $Sz(\Gamma)$ is given as in the following :

$$Sz(\Gamma) = \sum_{e \in E(\Gamma)} n_1(e|\Gamma) n_2(e|\Gamma),$$

where the summation embraces all edges of Γ,

$$n_1(e|\Gamma) = |\{v|v \in V(\Gamma), d(v, x|\Gamma) < d(v, y|\Gamma)\}|$$

and

$$n_2(e|\Gamma) = |\{v|v \in V(\Gamma), d(v, y|\Gamma) < d(v, x|\Gamma)\}|$$

which means that $n_1(e|\Gamma)$ counts the Γ's vertices are closer to one edge's terminal x than the other while $n_2(e|\Gamma)$ is vice versa [18].

[18] P.V. Khadikar, N.V. Deshpande, V. Narayan, P. Kale, P. Prabhakar, A. Dobrynin, I. Gutman, and G. Domotor, *The Szeged index and an analogy with the Wiener index, Journal of Chemical Information and Computer Sciences*, **35(3)** (1995), 547-550.

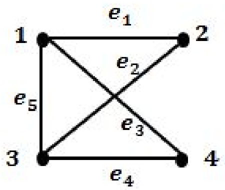


Figure 3: A simple connected graph

Note that $N_1(e_i|\Gamma)$ is the vertices of Γ lying closer to one endpoint x of the edge e_i than to its other endpoint y while $N_2(e_i|\Gamma)$ is vice versa. First, $N_1(e_i|\Gamma)$ and $N_2(e_i|\Gamma)$ are calculated for all i.
For $e_1 = \{1, 2\}$,

$$\begin{aligned} N_1(e_1|\Gamma) &= \{x \in V(\Gamma) : d(x,1) < d(x,2)\}, \quad n_1(e_1|\Gamma) = 2, \\ &= \{1, 4\}, \\ N_2(e_1|\Gamma) &= \{y \in V(\Gamma) : d(y,1) > d(y,2)\}, \quad n_2(e_1|\Gamma) = 1. \\ &= \{2\}, \end{aligned}$$

Example of Szeged Index (CONT.)

For $e_2 = \{2, 3\}$,

$$\begin{aligned} N_1(e_2|\Gamma) &= \{x \in V(\Gamma) : d(x,2) < d(x,3)\}, \quad n_1(e_2|\Gamma) = 1, \\ &= \{2\}, \\ N_2(e_2|\Gamma) &= \{y \in V(\Gamma) : d(y,2) > d(y,3)\}, \quad n_2(e_2|\Gamma) = 2. \\ &= \{3, 4\}, \end{aligned}$$

For $e_3 = \{1, 4\}$,

$$\begin{aligned} N_1(e_3|\Gamma) &= \{x \in V(\Gamma) : d(x,1) < d(x,4)\}, \quad n_1(e_3|\Gamma) = 2, \\ &= \{1, 2\}, \\ N_2(e_3|\Gamma) &= \{y \in V(\Gamma) : d(y,1) > d(y,4)\}, \quad n_2(e_3|\Gamma) = 1. \\ &= \{4\}, \end{aligned}$$

Example of Szeged Index (CONT.)

For $e_4 = \{3, 4\}$,

$$\begin{aligned} N_1(e_4|\Gamma) &= \{x \in V(\Gamma) : d(x,3) < d(x,4)\}, \quad n_1(e_4|\Gamma) = 2, \\ &= \{2, 3\}, \\ N_2(e_4|\Gamma) &= \{y \in V(\Gamma) : d(y,3) > d(y,4)\}, \quad n_2(e_4|\Gamma) = 1. \\ &= \{4\}, \end{aligned}$$

For $e_5 = \{1, 3\}$,

$$\begin{aligned} N_1(e_5|\Gamma) &= \{x \in V(\Gamma) : d(x,1) < d(x,3)\}, \quad n_1(e_5|\Gamma) = 1, \\ &= \{1\}, \\ N_2(e_5|\Gamma) &= \{y \in V(\Gamma) : d(y,1) > d(y,3)\}, \quad n_2(e_5|\Gamma) = 1. \\ &= \{3\}, \end{aligned}$$

Example of Szeged Index (CONT.)

Hence,

$$
\begin{aligned}
Sz(\Gamma) &= \sum_{i=1}^{5} n_1(e_i|\Gamma)n_2(e_i|\Gamma) \\
&= n_1(e_1|\Gamma)n_2(e_1|\Gamma) + n_1(e_2|\Gamma)n_2(e_2|\Gamma) + n_1(e_3|\Gamma)n_2(e_3|\Gamma) + \\
&\quad n_1(e_4|\Gamma)n_2(e_4|\Gamma) + n_1(e_5|\Gamma)n_2(e_5|\Gamma) \\
&= (2)(1) + (1)(2) + (2)(1) + (2)(1) + (1)(1) \\
&= 9.
\end{aligned}
$$

Definitions of Topological Indices

Harary Index

Let Γ be a connected graph with vertex set $V = \{1, 2, \ldots, n\}$. Half the elements' sum in the reciprocal distance matrix, $D^r = D^r(\Gamma)$, is what is known as the Harary index, written as

$$H = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D^r(i,j),$$

where

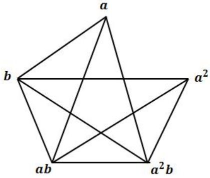
$$D^r(i,j) = \begin{cases} \frac{1}{d(i,j)} & \text{if} \quad i \neq j, \\ 0 & \text{if} \quad i = j, \end{cases}$$

and $d(i,j)$ is the shortest distance between vertex i and j [19].

[19] D. Plavšić, S. Nikolić, N. Trinajstić, and Z. Mihalić, *On the Harary index for the characterization of chemical graphs, Journal of Mathematical Chemistry*, **12** (1993), 235-250.

Example of Harary Index

Based on Figure 2,



$$H = \frac{1}{2}\sum_{i=a}^{a^2b}\sum_{j=a}^{a^2b} D^r(i,j)$$

$$= \frac{1}{2}\left[D^r(a,a) + D^r(a,a^2) + D^r(a,b) + D^r(a,ab) + D^r(a,a^2b) + \right.$$
$$D^r(a^2,a) + D^r(a^2,a^2) + D^r(a^2,b) + D^r(a^2,ab) + D^r(a^2,a^2b) +$$
$$D^r(b,a) + D^r(b,a^2) + D^r(b,b) + D^r(b,ab) + D^r(b,a^2b) +$$
$$D^r(ab,a) + D^r(ab,a^2) + D^r(ab,b) + D^r(ab,ab) + D^r(ab,a^2b) +$$
$$\left. D^r(a^2b,a) + D^r(a^2b,a^2) + D^r(a^2b,b) + D^r(a^2b,ab) + D^r(a^2b,a^2b)\right]$$

Example of Harary Index (CONT.)

$$H = \frac{1}{2}\sum_{i=a}^{a^2b}\sum_{j=a}^{a^2b} D^r(i,j)$$

$$= \frac{1}{2}\Big[0 + \frac{1}{2} + 1 + 1 + 1 + \frac{1}{2} + 0 + 1 + 1 + 1 +$$

$$1 + 1 + 0 + 1 + 1 + 1 + 1 + 1 + 0 + 1 + 1 + 1 + 1 + 1 + 0]$$

$$= 9.5$$

Definitions of Topological Indices

Randić Index

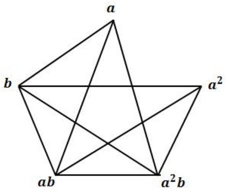
- The Randić index is a graph-theoretical descriptor that quantifies the complexity or branching structure of a molecular graph.
- It was introduced by Milan Randić [20] in 1975 and has found applications in various fields of chemistry.
- It is defined as the sum of the reciprocal square roots of the product of the degrees of connected pairs of vertices, written as

$$R(\Gamma) = \sum_{u,v \in E(\Gamma)} \frac{1}{\sqrt{deg(u)deg(v)}}.$$

[20] M. Randić , *Characterization of molecular branching. Journal of the American Chemical Society*, **97(23)** (1975), 6609-6615.

Example of Randić Index

Based on Figure 2,



$$R(\Gamma) = \sum_{u,v \in E(\Gamma)} \frac{1}{\sqrt{deg(u)deg(v)}}$$

$$= \frac{1}{\sqrt{(3)(4)}} + \frac{1}{\sqrt{(3)(4)}} + \frac{1}{\sqrt{(3)(4)}} + \frac{1}{\sqrt{(3)(4)}} + \frac{1}{\sqrt{(3)(4)}}$$

$$\frac{1}{\sqrt{(3)(4)}} + \frac{1}{\sqrt{(4)(4)}} + \frac{1}{\sqrt{(4)(4)}} + \frac{1}{\sqrt{(4)(4)}}$$

$$= 2.48$$

Definitions of Topological Indices

General Zeroth-Order Randić Index

The Randić index is modified and introduced a concept of general zeroth-order Randić index, which is defined as

$$^{0}R_{\alpha} = \sum_{u \in V(\Gamma)} (deg(u))^{\alpha},$$

where α can be any non-zero real number [21].

Sombor Index

Recently, in 2021, a new topological index, Sombor index has been established by Gutman [22]. The Sombor index of a graph, $SO(\Gamma)$, is defined as follows.

$$SO(\Gamma) = \sum_{u,v \in E(\Gamma)} \sqrt{\deg(u)^2 + \deg(v)^2}.$$

[21] H. Ahmed, A.A. Bhatti, and A. Ali, *Zeroth-order general Randić index of cactus graphs. AKCE International Journal of Graphs and Combinatorics*, **16(2)** (2019), 182-189.

[22] I. Gutman, *Geometric approach to degree-Based topological indices: Sombor indices. MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.

Example of Sombor Index

Based on Figure 2,

$$SO(\Gamma) = \sum_{u,v \in E(\Gamma)} \sqrt{\deg(u)^2 + \deg(v)^2}$$

$$= \sqrt{deg(a)^2 + deg(b)^2} + \sqrt{deg(a)^2 + deg(ab)^2} + \sqrt{deg(a)^2 + deg(a^2b)^2} +$$
$$\sqrt{deg(a^2)^2 + deg(b)^2} + \sqrt{deg(a^2)^2 + deg(ab)^2} +$$
$$\sqrt{deg(a^2)^2 + deg(a^2b)^2} + \sqrt{deg(b)^2 + deg(ab)^2} +$$
$$\sqrt{deg(b)^2 + deg(a^2b)^2} + \sqrt{deg(ab)^2 + deg(a^2b)^2}$$
$$= \sqrt{3^2 + 4^2} + \sqrt{3^2 + 4^2} + \sqrt{3^2 + 4^2} + \sqrt{3^2 + 4^2} + \sqrt{3^2 + 4^2}$$
$$\sqrt{3^2 + 4^2} + \sqrt{4^2 + 4^2} + \sqrt{4^2 + 4^2} + \sqrt{4^2 + 4^2}$$
$$= 46.97$$

The Coprime Graph [23]

Coprime graph of a group G is a graph that consists the elements in G as the set of vertices where two distinct vertices are adjacent if and only if the order of both vertices are coprime.

Example:

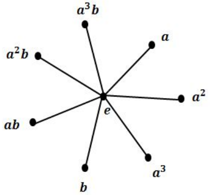


Figure 4: The coprime graph of D_8

[23] X.L. Ma, H.Q. Wei, and L.Y. Yang, *The coprime graph of a group. International Journal of Group Theory,* **3(3)** (2014), 13-23.

The Coprime Graph Associated to Dihedral Groups

In 2014, Ma *et al.* [23] generalized the coprime graph for certain order of dihedral groups, as stated in the following propositions.

Proposition 4 [23]

Let G be the dihedral groups of order $2n$ and the coprime graph of G is denoted as Γ_G^{CO}. Then, Γ_G^{CO} is isomorphic to a multipartite graph $K_{1,n-1,n}$ if n is an odd prime.

Proposition 5 [23]

Let G be the dihedral groups of order $2n$ and the coprime graph of G is denoted as Γ_G^{CO}. Then, Γ_G^{CO} is isomorphic to a star graph, $K_{1,2^{k+1}-1}$ if $n = 2^k$ for some positive integer k.

[23] X.L. Ma, H.Q. Wei, and L.Y. Yang, *The coprime graph of a group.* *International Journal of Group Theory,* **3(3)** (2014), 13-23.

Zero Divisor Graph [24]

Let R be a commutative ring with identity, $Z(R)$ its set of zero divisors. The zero divisor graph of R is $\Gamma(R) = Z(R) - 0$, with distinct vertices a and b adjacent if and only if $ab = 0$ or $ba = 0$.

Example :

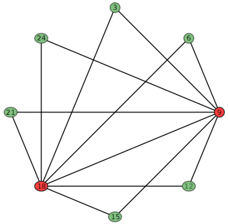


Figure 5: The zero divisor graph for $\mathbb{Z}_{27}$, $\Gamma\left(\mathbb{Z}_{27}\right)$

[24] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*. *Journal of Algebra*, **217(2)** (1999), 434–447.

TOPOLOGICAL INDICES OF GRAPHS ASSOCIATED TO GROUPS

The Wiener Index of the Non-commuting Graph for Some Finite Groups

Theorem 30 [25]

Let G be the dihedral groups, D_{2n} of order $2n$ where $n \geq 3$, Γ_G is the non-commuting graph of G and $W(\Gamma_G^{NC})$ is the Wiener index of Γ_G. Then,

$$W(\Gamma_G^{NC}) = \frac{1}{2}(5n^2 - 9n + 4).$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures*, Ph.D Thesis. Universiti Teknologi Malaysia (2021).

Proof

For n is odd and $n \geq 3$,

$$\begin{aligned}
W(\Gamma_G^{\mathrm{NC}}) &= \frac{1}{2}\left[(|G| - |Z(G)|)\left(|G| - 2|Z(G)| - 2\right) + |G|\left(k(G) - |Z(G)|\right)\right] \\
&= \frac{1}{2}\left[(2n-1)\left(2n - 2(1) - 2\right) + (2n)\left(\frac{n+3}{2} - 1\right)\right] \\
&= \frac{1}{2}\left[(2n-1)(2n-4) + n(n+1)\right] \\
&= \frac{1}{2}\left[4n^2 - 8n - 2n + 4 + n^2 + n\right] \\
&= \frac{1}{2}(5n^2 - 9n + 4).
\end{aligned}$$

Proof (Cont.)

For n is even and $n \geq 4$,

$$\begin{aligned} W(\Gamma_G^{\mathrm{NC}}) &= \frac{1}{2}\left[(|G|-|Z(G)|)\left(|G|-2|Z(G)|-2\right)+|G|\left(k(G)-|Z(G)|\right)\right] \\ &= \frac{1}{2}\left[(2n-2)\left(2n-2(2)-2\right)+(2n)\left(\frac{n+6}{2}-2\right)\right] \\ &= \frac{1}{2}\left[(2n-2)(2n-6)+n(n+2)\right] \\ &= \frac{1}{2}\left[4n^2-12n-4n+12+n^2+2n\right] \\ &= \frac{1}{2}\left[5n^2-14n+12\right]. \end{aligned}$$

■

The Wiener Index of the Non-commuting Graph for Some Finite Groups

Theorem 31 [26]

Let G be the generalised quaternion group, Q_{4n} of order $4n$ where $n \geq 2$, Γ_G is the non-commuting graph of G and $W(\Gamma_G^{NC})$ is the Wiener index of Γ_G. Then,

$$W(\Gamma_G^{NC}) = 2n(5n-7)+6.$$

Theorem 32 [25]

Let G be the quasidihedral group, QD_{2^n} of order 2^n where $n \geq 4$, Γ_G is the non-commuting graph of G and $W(\Gamma_G^{NC})$ is the Wiener index of Γ_G. Then,

$$W(\Gamma_G^{NC}) = 2^{2n-1} + 2^{2n-3} - 7(2^{n-1}) + 6.$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures*, Ph.D Thesis. Universiti Teknologi Malaysia (2021).
[26] N.H. Sarmin, N.I. Alimon, and A. Erfanian, *Topological indices of the non-commuting graph for generalised quaternion group. Bulletin of the Malaysian Mathematical Sciences Society*, **43(5)** (2020), 3361-3367.

The Wiener Index of the Coprime Graph for Some Finite Groups

The cases are only limited to n is an odd prime and $n = 2^k, k \in \mathbb{Z}$ since the coprime graph associated to the dihedral groups of the other cases of n cannot be generalized.

Theorem 33 [27]

Let G be the dihedral group, D_{2n} of order $2n$ where n is an odd prime. Then, the Wiener index of the coprime graph of G, Γ_G^{CO} is stated as follows :

$$W(\Gamma_G^{\mathrm{CO}}) = (n-1)(3n-1) + n.$$

Proof

- The coprime graph of D_{2n}, when n is an odd prime, is $K_{1,n-1,n}$.
- Then, the total number of vertices in $K_{1,n-1,n}$ is $1 + n - 1 + n = 2n$ vertices. Its coprime graph has three independent sets which are $1, n-1$ and n elements, respectively.

[27] N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Szeged and Wiener indices for coprime graph of dihedral groups. In AIP Conference Proceedings*, **2266(1)** (2020), 060006.

Proof (Cont.)

- The coprime graph of dihedral groups has n^2+n-1 edges where the number of edges for a complete graph, K_{2n} minus the number of edges for K_{n-1} and K_n, as shown in the following.

$$|E(K_{1,n-1,n})| = \frac{2n(2n-1)}{2} - \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2}$$
$$= n^2+n-1.$$

Thus, there are n^2+n-1 edges which have a distance of 1, while $\frac{(n-1)(n-2)}{2}$ and $\frac{n(n-1)}{2}$ edges have a distance of 2.
By using the definition of the Wiener index,

$$W(\Gamma_G^{\mathrm{CO}}) = \frac{1}{2}\sum_{i=1}^{2n}\sum_{j=1}^{2n} d(i,j)$$
$$= 1 \times \left[n^2+n-1\right] + 2 \times \left[\frac{n(n-1)}{2}\right] + 2 \times \left[\frac{(n-1)(n-2)}{2}\right]$$
$$= (n-1)(3n-1)+n.$$

The Wiener Index of the Coprime Graph for Some Finite Groups

Theorem 34 [27]

Let G be the dihedral group, D_{2n} of order $2n$ where $n = 2^k, k \in \mathbb{Z}^+$. Then, the Wiener index of the coprime graph for G,

$$W(\Gamma_G^{\mathrm{CO}}) = (2n-1)^2.$$

Theorem 35 [25]

Let G be the generalized quaternion group, Q_{4n} of order $4n$ where $n = 2^{k-1}, k \geq 2$ Then, the Wiener index of the coprime graph for G, Γ_G^{CO} is stated as follows :

$$W(\Gamma_G^{\mathrm{CO}}) = (4n-1)^2.$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures,* Ph.D Thesis. Universiti Teknologi Malaysia (2021).
[27] N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Szeged and Wiener indices for coprime graph of dihedral groups. In AIP Conference Proceedings*, **2266(1)** (2020), 060006.

The Wiener Index of the Coprime Graph for Some Finite Groups

Theorem 36 [25]

Let G be the quasidihedral group, Q_{2^n} of order 2^n where $n \geq 4$. Then, the Wiener index of the coprime graph for G, Γ_G^{CO} is stated as follows :

$$W(\Gamma_G^{\mathrm{CO}}) = (2^n - 1)^2.$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures,* Ph.D Thesis. Universiti Teknologi Malaysia (2021).

The Zagreb Index of the Non-commuting Graph for Some Finite Groups

Theorem 37 [28]

Let G be a dihedral group, D_{2n} of order $2n$ where $n \geq 3$. Then,

$$M_1(\Gamma_G^{NC}) = \begin{cases} n(5n-4)(n-1) & \text{if} \quad n \text{ is odd,} \\ n(5n-8)(n-2) & \text{if} \quad n \text{ is even.} \end{cases}$$

[28] N.I. Alimon, N.H., Sarmin, and A. Erfanian, *Topological indices of non-commuting graph of dihedral groups. Malaysian Journal of Fundamental and Applied Sciences*, (2018), 473-476.

Proof

For n is odd,

$$\begin{aligned} M_1(\Gamma_G^{\mathrm{NC}}) &= |G|^2\left(|G| + |Z(G)| - 2k(G)\right) - \sum_{x \in G - Z(G)} |C_G(x)|^2 \\ &= 4n^2\left[2n + 1 - 2\left(\frac{n+3}{2}\right)\right] - 2^2 n + n^2(n-1) \\ &= n\left(5n-4\right)\left(n-1\right). \end{aligned}$$

For n is even,

$$\begin{aligned} M_1(\Gamma_G^{\mathrm{NC}}) &= |G|^2\left(|G| + |Z(G)| - 2k(G)\right) - \sum_{x \in G - Z(G)} |C_G(x)|^2 \\ &= 4n^2\left[2n + 1 - 2\left(\frac{n+6}{2}\right)\right] - 4^2 n + n^2(n-2) \\ &= n\left(5n-8\right)\left(n-2\right). \end{aligned}$$

■

The Zagreb Index of the Non-commuting Graph for Some Finite Groups

Theorem 38 [28]

Let G be a dihedral group, D_{2n} of order $2n$ where $n \geq 3$. Then,

$$M_2(\Gamma_G^{NC}) = \begin{cases} 2n(n-1)^2(2n-1) & \text{if} \quad n \text{ is odd}, \\ 4n(n-2)^2(n-1) & \text{if} \quad n \text{ is even}. \end{cases}$$

Theorem 39 [26]

Let G be the generalised quaternion group, Q_{4n} of order $4n$ where $n \geq 2$. Then,

$$M_1(\Gamma_G^{NC}) = 8n(5n^2 - 9n + 4),$$

and

$$M_2(\Gamma_G^{NC}) = 32n(2n^3 - 5n^2 + 4n - 1).$$

[28] N.I. Alimon, N.H., Sarmin, and A. Erfanian, *Topological indices of non-commuting graph of dihedral groups. Malaysian Journal of Fundamental and Applied Sciences*, (2018), 473-476.
[26] N.H. Sarmin, N.I. Alimon, and A. Erfanian, *Topological indices of the non-commuting graph for generalised quaternion group. Bulletin of the Malaysian Mathematical Sciences Society*, **43(5)** (2020), 3361-3367.

The Zagreb Index of the Non-commuting Graph for Some Finite Groups

Theorem 40 [25]

Let G be the quasidihedral group, QD_{2^n} of order 2^n where $n \geq 4$. Then,

$$M_1(\Gamma_G^{\mathrm{NC}}) = \left[5(2^{3n-3}) - 9(2^{2n-1}) + 8(2^n)\right],$$

and

$$M_2(\Gamma_G^{\mathrm{NC}}) = \left[2^{4n-2} - 5(2^{3n-1}) + 8(3^n) - 8(2^n)\right].$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures*, Ph.D Thesis. Universiti Teknologi Malaysia (2021).

The Szeged Index of the Non-commuting Graph for D_{2n}

Theorem 41 [29]

Let G be the dihedral groups, D_{2n}, where $n \geq 3$. Then, the Szeged index of the non-commuting graph of G,

$$Sz(\Gamma_G^{NC}) = \begin{cases} n(n-1)(n-\frac{1}{2}), & \text{if} \quad n \,\text{is odd}, \\ 2n(n-2)(n-1), & \text{if} \quad n \,\text{is even}. \end{cases}$$

Proof

By Proposition 1, the non-commuting graph of D_{2n} is

$$\Gamma_G^{NC} = \begin{cases} K_{\underbrace{1,1,\ldots,1}_{\text{n times}},n-1}, & \text{if} \quad n \text{ is odd}, \\ K_{\underbrace{2,2,\ldots,2}_{\frac{n}{2}\ \text{times}},n-2}, & \text{if} \quad n \text{ is even}. \end{cases}$$

[29] N.I. Alimon, N.H., Sarmin, and A. Erfanian, *On the Szeged index and its non-commuting graph, Jurnal Teknologi*, **85(3)** (2023), 105-110.

Proof (Cont.)

For n is odd and $n \geq 3$:

- There are $n(n-1)$ edges have $n_1(e|\Gamma) = 1$ since there is only one element of vertices which is closer to a vertex a^i of the edge than the other vertex of the edge, $a^j b$, where $i = \{1, 2, \ldots, n-1\}$ and $j = \{0, 1, \ldots, n-1\}$.
- Then, $n_2(e|\Gamma) = n-1$ since $n-1$ elements of vertices which are closer to $a^k b$ than to $a^l b$, where $k, l = \{0, 1, \ldots, n-1\}$. Meanwhile, the rest of the edges have $n_1(e|\Gamma) = n_2(e|\Gamma) = 1$.

Thus, by definition of the Szeged index,

$$
\begin{aligned}
Sz(\Gamma_G^{\mathrm{NC}}) &= n(n-1)[1 \times (n-1)] + (|E(\Gamma_G)| - n(n-1))[1 \times 1] \\
&= n(n-1)(n-1) + \left[\frac{|G|^2 - k(G)|G|}{2} - n(n-1)\right] \\
&= n(n-1)(n-1) + [2n^2 - n\frac{n+3}{2} - n(n-1)] \\
&= n(n-1)^2 + \frac{n^2}{2} - \frac{n}{2} \\
&= n(n-1)(n - \frac{1}{2}).
\end{aligned}
$$

Proof (Cont.)

For n is even and $n \geq 3$:

- There are $2n(n-2)$ edges have $n_1(e|\Gamma) = 2$ since there are two elements of vertices which are closer to a vertex of edge, a^i than the other vertex of edge, $a^j b$, where a^i is non-central elements and $j = \{0, 1, \dots, n-1\}$.

Then, $n_2(e|\Gamma) = n-2$ since there is $n-2$ elements of vertices which re closer to $a^k b$ than to $a^l b$, where $k, l = \{0, 1, \dots, n-1\}$. Meanwhile, the rest of the edges have $n_1(e|\Gamma) = n_2(e|\Gamma) = 2$. Thus, by the definition of Szeged index:

$$\begin{aligned} Sz(\Gamma_G) &= n(n-2)[2 \times (n-2)] + (|E(\Gamma_G)| - n(n-2))[2 \times 2] \\ &= 2n(n-2)(n-2) + \left[\frac{|G|^2 - k(G)|G|}{2} - n(n-2)\right][2 \times 2] \\ &= 2n(n-2)^2 + \left[\frac{4n^2 - (n+6)(n)}{2} - n(n-2)\right][2 \times 2] \\ &= 2n(n-2)^2 + 2[4n^2 - n(n+6) - 2n(n-2)] \\ &= 2n(n-2)^2 + 2n(n-2) \\ &= 2n(n-2)(n-1). \end{aligned}$$

∎

The Szeged Index of the Non-commuting Graph for Q_{4n} and QD_{2^n}

Theorem 42 [29]

Let G be the generalized quaternion groups, Q_{4n} where $n \geq 2$. Then, the Szeged index of the non-commuting graph of G,

$$Sz(\Gamma_G^{\mathrm{NC}}) = 8n(2n-1)(n-1).$$

Theorem 43 [29]

Let G be the quasidihedral groups of order 2^n where $n \geq 4$. Then, the Szeged index of the non-commuting graph of G,

$$Sz(\Gamma_G^{\mathrm{NC}}) = \left[2^{3n-2} - 3(2^{2n-1}) + 2^{n+1}\right].$$

[29] N.I. Alimon, N.H., Sarmin, and A. Erfanian, *On the Szeged index and its non-commuting graph, Jurnal Teknologi,* **85(3)** (2023), 105-110.

The Szeged Index of the Coprime Graph for D_{2n}

Theorem 44 [27]

Let G be the dihedral groups, D_{2n} where $n \geq 3$. If n is an odd prime, then the Szeged index of the coprime graph of G,

$$Sz(\Gamma_G^{\mathrm{CO}}) = n^4 - 2n^3 + 3n^2 - 2n + 1.$$

Theorem 45 [27]

Let G be a dihedral group, D_{2n} and Γ_G^{CO} is coprime graph of G. Then, if $n = 2^k$, where $k \in \mathbb{Z}^+$, the Szeged index of coprime graph for D_{2n} is as follows :

$$Sz(\Gamma_G^{CO}) = 4n^2 - 4n + 1.$$

[27] N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Szeged and Wiener indices for coprime graph of dihedral groups. In AIP Conference Proceedings*, **2266(1)** (2020), 060006.

The Szeged Index of the Coprime Graph for Q_{4n} and QD_{2^n}

Theorem 46 [25]

Let G be the generalized quaternion groups of order $4n$ where $n \geq 2$. If $n = 2^{k-1}, k \geq 2$, then the Szeged index of the coprime graph of G,

$$Sz(\Gamma_G^{CO}) = (4n-1)^2.$$

Theorem 48 [25]

Let G be the quasidihedral groups of order 2^n where $n \geq 4$. Then, the Szeged index of the coprime graph of G,

$$Sz(\Gamma_G^{CO}) = (2^n-1)^2.$$

[25] N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures,* Ph.D Thesis. Universiti Teknologi Malaysia (2021).

The Harary Index of the Non-commuting Graph for D_{2n}

Theorem 49 [30]

Let G be a dihedral group, D_{2n} and Γ_G^{NC} is a non-commuting graph of G. Then,

$$H(\Gamma_G^{NC}) = \begin{cases} \frac{1}{4}\left[(n-2)(7n-3)+n\right] & \text{if} \quad n \text{ is even,} \\ \frac{1}{4}\left[(n-1)(7n-2)\right] & \text{if} \quad n \text{ is odd.} \end{cases}$$

[30] N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Harary index of the non-commuting graph for dihedral groups. Southeast Asian Bull. Math*, **44(6)** (2020), 763-768.

Proof

The Harary index of the non-commuting graph for dihedral group where n is odd is the same as half of the total of all entries in its distance matrix, D^r. The entries of D^r in this case is either 1 if two vertices are connected or $\frac{1}{2}$ if two vertices are not connected to each other. Hence,
for n is even and $n \geq 4$,

$$\begin{aligned}
H(\Gamma_G^{NC}) &= |E(\Gamma_G^{NC})| + \left(\frac{1}{2} \times \frac{n}{2}\right) + \left(\frac{(n-2)(n-3)}{2} \times \frac{1}{2}\right) \\
&= \frac{3}{2}n(n-2) + \frac{n}{4} + \frac{(n-2)(n-3)}{4} \\
&= \frac{1}{4}\left[6n(n-2) + n + (n-2)(n-3)\right] \\
&= \frac{1}{4}\left[(n-2)(7n-3) + n\right],
\end{aligned}$$

Proof (Cont.)

for n is odd and $n \geq 3$,

$$\begin{aligned} H(\Gamma_G^{NC}) &= |E(\Gamma_G^{NC})| + \frac{1}{2}\left[\frac{(n-1)(n-2)}{2}\right] \\ &= \frac{3n}{2}(n-1) + \frac{1}{4}(n-1)(n-2) \\ &= \frac{1}{2}(n-1)\left(3n + \frac{1}{2}(n-2)\right) \\ &= \frac{1}{2}(n-1)\left(\frac{7}{2}n - 1\right) \\ &= \frac{1}{4}(n-1)(7n-2). \end{aligned}$$

Therefore,

$$H(\Gamma_G^{NC}) = \begin{cases} \frac{1}{4}[(n-2)(7n-3)+n], & \text{if} \quad n \text{ is even}, \\ \frac{1}{4}(n-1)(7n-2), & \text{if} \quad n \text{ is odd}. \end{cases} \qquad \blacksquare$$

The Harary Index of the Non-commuting Graph for Q_{4n} and QD_{2^n}

Theorem 50 [30]

Let G be the generalized quaternion groups of order $4n$, where $n \geq 2$. Then, the Harary index of the non-commuting graph of G,

$$H(\Gamma_G^{\mathrm{NC}}) = 7n^2 - 8n + \frac{3}{2}.$$

Theorem 51 [30]

Let G be the quasidihedral groups of order 2^n, where $n \geq 4$. Then, the Harary index of the non-commuting graph of G,

$$H(\Gamma_G^{\mathrm{NC}}) = 7(2^{2n-4}) - 2^{n+1} + \frac{3}{2}.$$

[30] N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Harary index of the non-commuting graph for dihedral groups. Southeast Asian Bull. Math*, **44(6)** (2020), 763-768.

The Randić Index of the Non-commuting Graph for Some Finite Groups

Theorem 52 [31]

Let G be a dihedral group, D_{2n} and Γ_G^{NC} is a non-commuting graph of G. Then,

$$R(\Gamma_G^{NC}) = \begin{cases} \frac{n\sqrt{2n(n-1)}+4n(n-1)}{4\sqrt{2n(n-1)}} & \text{if} \quad n \text{ is odd,} \\ \frac{n\sqrt{2n(n-2)}+4n(n-2)}{4\sqrt{2n(n-2)}} & \text{if} \quad n \text{ is even.} \end{cases}$$

[31] S.R.D. Roslly, N.F.A.Z. Ab Halem, N.S.S. Zailani, and N.I. Alimon, *Generalization of Randić index of the non-commuting graph for a family of finite groups. Malaysian Journal of Fundamental and Applied Sciences*, Malaysian Journal of Fundamental and Applied Sciences, **19(5)** (2023), 762-768.

The Randić Index of the Non-commuting Graph for Some Finite Groups

Theorem 53 [31]

Let G be the generalised quaternion group, Q_{4n} and Γ_G^{NC} is a non-commuting graph of G. Then,

$$R(\Gamma_G^{NC}) = \frac{4n(n-1)}{\sqrt{8n(n-1)}} + \frac{n}{2}.$$

Theorem 54 [31]

Let G be the quasidihedral group, QD_{2^n} and Γ_G^{NC} is a non-commuting graph of G. Then,

$$R(\Gamma_G^{NC}) = \frac{2^{n-1}(2^{n-1}-2)}{\sqrt{(2^{n-1})(2^n-4)}} + \frac{2^{n-1}(2^{n-2}-1)}{2^n-4}.$$

[31] S.R.D. Roslly, N.F.A.Z. Ab Halem, N.S.S. Zailani, and N.I. Alimon, *Generalization of Randić index of the non-commuting graph for a family of finite groups. Malaysian Journal of Fundamental and Applied Sciences*, Malaysian Journal of Fundamental and Applied Sciences, **19(5)** (2023), 762-768.

The Sombor Index of the Non-commuting Graph for Some Finite Groups

Theorem 55 [32]

Let Γ_G be the non-commuting graph of G where G is the dihedral groups of order $2n$, $n \geq 3$. Then, the Sombor index of Γ_G^{NC},

$$SO(\Gamma_G^{NC}) = \begin{cases} (n-1)[\sqrt{2}(n-1) + \sqrt{4(n-1)^2 + n^2}] & \text{if} \quad n \text{ is odd}, \\ (n-2)[\sqrt{2}(n-2) + \sqrt{4(n-2)^2 + n^2}] & \text{if} \quad n \text{ is even}. \end{cases}$$

Theorem 56 [32]

Let Γ_G be the non-commuting graph of G where G is the generalized quaternion groups of order $4n$, $n \geq 2$. Then, the Sombor index of Γ_G^{NC},

$$SO(\Gamma_G^{NC}) = n(n-1)[\sqrt{2}(n-1) + \sqrt{4(n-1)^2 + n^2}].$$

[32] S.M.S. Khasraw, N.H. Sarmin, N.I. Alimon, N. Najmuddin, and G. Semil @ Ismail, *The Sombor index and Sombor polynomial of the power graph associated to some finite groups*, *Journal of Advanced Research in Applied Sciences and Engineering Technology*, **42(2)**, (2024), 112-121.

The Sombor Index of the Non-commuting Graph for Some Finite Groups

Theorem 57 [32]

Let Γ_G be the non-commuting graph of G where G is the quasidihedral groups of order 2^n, $n \geq 4$. Then, the Sombor index of Γ_G^{NC},

$$SO(\Gamma_G^{NC}) = \sqrt{2}(2^n - 4)\left(2^{2n-3} - 2^{n+1}\right) + \left(2^{2n-2} - 2\right)\sqrt{(2^n - 4)^2 + 2^{2n-2}}.$$

[32] S.M.S. Khasraw, N.H. Sarmin, N.I. Alimon, N. Najmuddin, and G. Semil @ Ismail, *The Sombor index and Sombor polynomial of the power graph associated to some finite groups*, *Journal of Advanced Research in Applied Sciences and Engineering Technology*, **42(2)**, (2024), 112-121.

TOPOLOGICAL INDICES OF GRAPHS ASSOCIATED TO RINGS

The First Zagreb Index of the Zero Divisor Graph for the Ring $\mathbb{Z}_{p^k}$

Proposition 6 [35]

Let p be a prime number, $k \in \mathbb{N}$ and $a \in \mathbb{Z}_{p^k}$ with $\gcd(a, p^k) = p^i$ for $i = 1, 2, \ldots, k$. Then, the degree of vertex a of the zero divisor graph for the ring $\mathbb{Z}_{p^k}$ is

$$\deg(a) = \begin{cases} p^i - 1, & \text{for } i \leq \left\lfloor \frac{k-1}{2} \right\rfloor, \\ p^i - 2, & \text{for } i > \left\lfloor \frac{k-1}{2} \right\rfloor, \end{cases}$$

where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p, and $\lfloor k \rfloor$ denotes the floor function of k.

Proposition 7 [35]

Let $a \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$ thus $a \in Z\left(\mathbb{Z}_{p^k}\right)$ where $\gcd(a, p^k) = p^i$. Then $\left|V\left(\mathbb{Z}_{p^k}\right)\right| = p^{k-i} - p^{k-(i+1)}$ for $1 \leq i \leq k-1$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p.

[35] G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *The first Zagreb index of zero divisor graph for the ring of integers modulo power of primes, Malaysian Journal of Fundamental and Applied Sciences*, **19(5)** (2023), 892-900.

The First Zagreb Index of the Zero Divisor Graph for the Ring $\mathbb{Z}_{p^k}$

Theorem 58 [35]

The first Zagreb index of the zero divisor graph for the ring $\mathbb{Z}_{p^k}$,

$M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = 2(p^{k-1} - p^k)\left(k - 1 + \lceil\frac{k-1}{2}\rceil\right) + (p^k + 1)(p^{k-1} - 1) + 3\left(p^{\lceil\frac{k-1}{2}\rceil} - 1\right)$

where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p.

Proof.

Using definition of the first Zagreb index, Proposition 6, and Proposition 7,

$$M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \sum_{u \in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)} \left(deg\left(u\right)\right)^2$$

$$= \sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor} \left(p^{k-i} - p^{k-(i+1)}\right)\left(p^i - 1\right)^2 + \sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1} \left(p^{k-i} - p^{k-(i+1)}\right)$$

$$\left(p^i - 2\right)^2$$

Proof.

$$= \sum_{i=1}^{k-1} \left(p^{k-i} - p^{k-(i+1)}\right) \left(p^{2i} - 2p^{i} + 1\right) + \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1} \left(p^{k-i} - p^{k-(i+1)}\right) \left(3 - 2p^{i}\right).$$

Using the summation rules and the geometric sequences,

$$M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = 2\left(p^{k-1} - p^{k}\right) \sum_{i=1}^{k-1} 1 + \ p^{k} \sum_{i=1}^{k-1} \left(p^{i} + \frac{1}{p^{i}} - p^{i-1} - \frac{1}{p^{i+1}}\right) + 2\left(p^{k-1} - p^{k}\right) \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1} 1 + \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1} \left(\frac{1}{p^{i}} - \frac{1}{p^{i+1}}\right).$$

The First Zagreb Index of the Zero Divisor Graph for the Ring $\mathbb{Z}_{p^k}$

Proof.

Therefore, the first Zagreb index of the zero divisor graph for the ring $\mathbb{Z}_{p^k}$,

$$M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = 2\left(p^{k-1} - p^k\right)\left(k - 1 + \left\lceil \frac{k-1}{2} \right\rceil\right) + \left(p^k + 1\right)\left(p^{k-1} - 1\right) + 3\left(p^{\left\lceil \frac{k-1}{2} \right\rceil} - 1\right),$$

where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p. □

[35] G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *The first Zagreb index of zero divisor graph for the ring of integers modulo power of primes*, *Malaysian Journal of Fundamental and Applied Sciences*, **19(5)** (2023), 892-900.

The First Zagreb Index of the Zero Divisor Graph for the Ring $\mathbb{Z}_{2^k q}$

Theorem 59 [36]

The first Zagreb index of the zero divisor graph for the ring $\mathbb{Z}_{2^k q}$, $M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = (q-1)\left[\left(2^k - 2^{k-1}\right)(q-2k+1) + \left(2^k-1\right)\left(2^k+2^{k-1}-1\right)-1\right] + q^2 2^k\left(2^{k-2}-1\right) + \left(2^{k-1}q-2\right)^2 - 2q\left(2^k-2^{k-1}\right)\left(k-2+\left\lceil\frac{k-3}{2}\right\rceil\right) - 2 + 2^{k-1} + 3\left(\frac{2^k - 2^{\left\lfloor\frac{k+3}{2}\right\rfloor}}{2^{\left\lfloor\frac{k+1}{2}\right\rfloor}}\right)$ where q is an odd prime number and k is a positive integer.

[36] G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *The First Zagreb Index of the Zero Divisor Graph for the Ring of Integers Modulo* $2^k q$, AIP Conf. Proc., In press.

The General Zeroth-Order Randić Index of the Zero Divisor Graph for $\mathbb{Z}_{p^k}$

Theorem 60 [37]

The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^k}$ when $\alpha = 1$, $R_1^0\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \left(p^k - p^{k-1}\right)(k-1) - p^{\left\lceil \frac{k-1}{2} \right\rceil} - p^{k-1} + 2$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd prime p.

Notice that the general zeroth-order Randić index of the zero divisor graph for the ring $\mathbb{Z}_{p^k}$ when $\alpha = 2$ is equal to the first Zagreb index of the zero divisor graph for the ring $\mathbb{Z}_{p^k}$, $R_2^0\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)$, as shown in the following theorem.

Theorem 61 [37]

The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^k}$ when $\alpha = 2$, $R_2^0\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = 2\left(p^{k-1} - p^k\right)\left(k - 1 - \left\lceil \frac{k-1}{2} \right\rceil\right)$ $+\mathrm{p}^{2k-1} + p^{k-1} - p^k + 3\left(p^{\left\lceil \frac{k-1}{2} \right\rceil} - 1\right) - 1$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd prime p.

[37] G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *General zeroth-order Randić index of zero divisor graph for the ring of integers modulo p^n*, AIP Conf. Proc. **2975**, 020002 (2023).

The General Zeroth-Order Randić Index of the Zero Divisor Graph for $\mathbb{Z}_{p^k}$

Theorem 62 [37]

The general zeroth-order Randić index of the zero divisor graph for $\mathbb{Z}_{p^k}$ when $\alpha = 3$, $R_3^0\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \frac{p^{3k-1} - p^{k+1}}{p+1} + 3\left(p^k - p^{k-1}\right)\left(k - 1 + 3\left\lceil\frac{k-1}{2}\right\rceil\right) - 6p^{2k-1} + 3p^k - p^{k-1} + 3p^{\left\lfloor\frac{3k-1}{2}\right\rfloor} - 7p^{\left\lfloor\frac{k}{2}\right\rfloor} + 8$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd prime p.

[37] G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *General zeroth-order Randić index of zero divisor graph for the ring of integers modulo* p^n, AIP Conf. Proc. **2975**, 020002 (2023).

The General Zeroth-Order Randić Index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})$

Theorem 63 [38]

The general zeroth-order Randić index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})$, $R_\alpha^0\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})\right) =$
$(q-1)(pq-2)^\alpha + (p-1)\left(q^2-1\right)^\alpha + (q^2-q)(p-1)^\alpha + (pq-p-q+1)(q-1)^\alpha$
where p, q are primes and $\alpha \in \mathbb{R}$.

[38] Nurhabibah, A.G. Syarifudin, I.M. Alamsyah, E. Suwastika, N.H. Sarmin, N.I. Alimon, and G. Semil @ Ismail, *Topological Indices of the Zero Divisor Graph of Direct Product of Integers Modulo Ring*, AIP Conf. Proc., Submitted.

Conclusion

- The general formulas for energy of conjugacy class graphs related to dihedral groups D_{2n}, generalized quaternion groups Q_{4n} and quasidihedral groups QD_{2^n} have been found.
- The eigenvalues of the non-commuting graphs related to dihedral groups D_{2n}, generalized quaternion groups Q_{4n} and quasidihedral groups QD_{2^n} have been found and were then used to obtain the general formulas of the energy of non-commuting graphs associated to these groups.
- The energy of the Cayley graphs associated to the dihedral groups D_{2n}, alternating groups A_n and symmetric groups S_n with respect to subsets of order one and two have been computed.
- Moreover, the Seidel energy of the Cayley graphs associated to the dihedral groups D_{2n}, alternating groups A_n and symmetric groups S_n with respect to subsets of order one and two have computed.
- Some topological indices, which are the Wiener index, the Zagreb index, the Szeged index, the Harary index and the Randić index of some graphs associated to some finite groups are found.

Conclusion

- Based on the results, the higher the order of the groups, the higher the value of topological indices. This is due to the increasing of the number of vertices and edges.
- The degree-based topological indices, which are the first Zagreb index and the general zeroth-order Randić index of the zero divisor graph for the rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{2^k q}$ are determined.

Future Research Recommendations

- Algorithms and techniques to obtain the adjacency matrices and the Laplacian matrices of these graphs related to D_{2n}, Q_{4n} and QD_{2^n}, can be developed for any integer n.
- Since this research has provide some useful knowledge such as the connectivities of the Cayley graphs and the graph theoretical properties such as their spectrums and their energies, they can be developed into various codes and algorithms to be used in computer engineering students.
- Finding the graphs associated to the relative commutativity degree of subgroups of dihedral groups for other cases. From there, the energy for more cases of dihedral groups D_{2n} can be found.
- Other types of topological indices of some graphs associated to groups can be computed.
- Similar to other types of topological indices of some graphs associated to rings can also be computed.
- The topological indices for graphs representing chemical structures in drugs are a valuable approach to determine both the physicochemical properties and biological activities of these molecules can be constructed.

- I. Gutman, *The energy of a graph, Der. Math. stat. Sekt. Forschungszent*, (1978) 1-22.
- I. Gutman and B. Zhou, *Laplacian energy of a graph, Linear Algebra and its Applications*, **414(1)** (2006), 29-37.
- R.B. Bapat, *Graphs and matrices, Springer*, **27** (2010).
- J. P. Liu, , and B. L. Liu, *Generalization for Laplacian energy, Applied Mathematics-A Journal of Chinese Universities*, **24(4)** (2009), 443.
- W. H. Haemers, *Seidel switching and graph energy*, Available at SSRN 2026916, (2012).
- A. Abdollahi, S. Akbari, H. Maimani, *Non-commuting graph of a group. Journal of Algebra.* **298** (2006), 468-492.
- E. A. Bertram, M. Herzog, and A. Mann, *On a graph related to conjugacy classes of groups, Bulletin of the London Mathematical Society*, **22**(6) (1990), 569–575.
- L. W. Beineke, and R. J. Wilson, *Topics in algebraic graph theory*, **102**,USA : Cambridge University Press, (2004).
- J. Humphreys, *A course in group theory, vol. 6, Oxford University Press on Demand*, 1996.
- R. Mahmoud, *Energy and Laplacian Energy of Graphs Related to a Family of Finite Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2018.

A. F. Ahmad Fadzil, *The Energy and Seidel Energy of Cayley Graphs Associated to Dihedral, Alternating and Symmetric Groups*, Ph.D Thesis, Universiti Teknologi Malaysia. 2021.

I. Gutman and O.E. Polansky, *Mathematical Concepts in Organic Chemistry, Springer Science and Business Media*, 2012.

K. Wiener, *Structural determination of paraffin boiling points, Journal of the American Chemical Society*, **69(1)** (1947), 17-20.

H. Hosoya, *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bulletin of the Chemical Society of Japan*, **44(9)** (1971), 2332–2339.

A. Azad and M. Eliasi, *Distance in the non-commuting graph of groups. Ars Comb.* **99** (2011), 279-287.

I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total* π*-electron energy of alternant hydrocarbons, Chemical Physics Letters*, **17(4)** (1972), 535-538.

M. Mizargar and A. Ashrafi, *Some distance-based topological indices of a non-commuting graph. Hacettepe Journal of Mathematics ans Staristics.* **41(4)** (2012), 515-526.

P.V. Khadikar, N.V. Deshpande, V. Narayan, P. Kale, P. Prabhakar, A. Dobrynin, I. Gutman, and G. Domotor, *The Szeged index and an analogy with the Wiener index, Journal of Chemical Information and Computer Sciences*, **35(3)** (1995), 547-550.

D. Plavšić, S. Nikolić, N. Trinajstić, and Z. Mihalić, *On the Harary index for the characterization of chemical graphs, Journal of Mathematical Chemistry*, **12** (1993), 235-250.

REFERENCES III

M. Randić, *Characterization of molecular branching. Journal of the American Chemical Society*, **97(23)** (1975), 6609-6615.

H. Ahmed, A.A. Bhatti, and A. Ali, *Zeroth-order general Randić index of cactus graphs. AKCE International Journal of Graphs and Combinatorics*, **16(2)** (2019), 182-189.

I. Gutman, *Geometric approach to degree-Based topological indices: Sombor indices. MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.

X.L. Ma, H.Q. Wei, and L.Y. Yang, *The coprime graph of a group. International Journal of Group Theory,* **3(3)** (2014), 13-23.

D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring. Journal of Algebra,* **217(2)** (1999), 434–447.

N.I. Alimon, *Topological Indices of a Class of Graphs of Some Finite Groups and Applications to Molecular Structures,* Ph.D Thesis. Universiti Teknologi MAlaysia (2021).

N.H. Sarmin, N.I. Alimon, and A. Erfanian, *Topological indices of the non-commuting graph for generalised quaternion group. Bulletin of the Malaysian Mathematical Sciences Society*, **43(5)** (2020), 3361-3367.

N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Szeged and Wiener indices for coprime graph of dihedral groups. In AIP Conference Proceedings*, **2266(1)** (2020), 060006.

N.I. Alimon, N.H., Sarmin, and A. Erfanian, *Topological indices of non-commuting graph of dihedral groups. Malaysian Journal of Fundamental and Applied Sciences*, (2018), 473-476.

REFERENCES IV

- N.I. Alimon, N.H., Sarmin, and A. Erfanian, *On the Szeged index and its non-commuting graph, Jurnal Teknologi*, **85(3)** (2023), 105-110.
- N.I. Alimon, N.H. Sarmin, and A. Erfanian, *The Harary index of the non-commuting graph for dihedral groups. Southeast Asian Bull. Math*, **44(6)** (2020), 763-768.
- S.R.D. Roslly, N.F.A.Z. Ab Halem, N.S.S. Zailani, and N.I. Alimon, *Generalization of Randić index of the non-commuting graph for a family of finite groups. Malaysian Journal of Fundamental and Applied Sciences*, **19(5)** (2023), 762-768.
- S.M.S. Khasraw, N.H. Sarmin, N.I. Alimon, N. Najmuddin, and G. Semil @ Ismail *The Sombor index and Sombor polynomial of the noncommuting graph associated to some finite groups, Journal of Advanced Research in Applied Sciences and Engineering Technology*, **42(2)** (2024), 112-121.
- J.B. Fraleigh, *A first course in abstract algebra, Pearson Education India*, 2003.
- R. Wisbauer, *Foundations of module and ring theory, Routledge*, 2018.
- G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *The first Zagreb index of zero divisor graph for the ring of integers modulo power of primes, Malaysian Journal of Fundamental and Applied Sciences*, **19(5)** (2023), 892-900.
- G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *The First Zagreb Index of the Zero Divisor Graph for the Ring of Integers Modulo* $2^k q$, AIP Conf. Proc., In press.

G. Semil @ Ismail, N.H. Sarmin, N.I. Alimon, and F. Maulana, *General zeroth-order Randić index of the zero divisor graph for some commutative rings*, AIP Conf. Proc. **2905**, 070005 (2024).

Nurhabibah, A.G. Syarifudin, I.M. Alamsyah, E. Suwastika, N.H. Sarmin, N.I. Alimon, and G. Semil @ Ismail, *Topological Indices of the Zero Divisor Graph of Direct Product of Integers Modulo Ring*, AIP Conf. Proc., Submitted.

PUBLICATION

https://people.utm.my/nizasarmin/journal-papers/

https://people.utm.my/nizasarmin/conference-proceedings/

DR. RABIHA MAHMOUD BIRKIA

1. Rabiha Mahmoud, Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin, Ahmad Erfanian. "The Laplacian Energy of the Conjugacy Class Graph of Some Finite Groups", MATEMATIKA, 35(1), 59-65, 2019.
2. Rabiha Mahmoud, Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin, Ahmad Erfanian. "The Laplacian Energy of Conjugacy Class Graph of Some Dihedral Groups", Malaysian Journal of Fundamental and Applied Sciences, 13(2), 129-131, 2017.
3. Rabiha Mahmoud, Nor Haniza Sarmin, Ahmad Erfanian, "The conjugacy class graph of some finite groups and its energy", Malaysian Journal of Fundamental and Applied Sciences, 13(4), 659-665, 2017.
4. Rabiha Mahmoud, Nor Haniza Sarmin and Ahmad Erfanian, "On the energy of non-commuting graph of dihedral groups", AIP Conference Proceedings 1830, 070011 (2017).

DR. AMIRA FADINA AHMAD FADZIL

1 Nor Haniza Sarmin, Amira Fadina Ahmad Fadzil and Ahmad Erfanian, "Seidel energy for Cayley graphs associated to dihedral group", Journal of Physics: Conference Series, 1988: 012066, 2021.

2 Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin and Ahmad Erfanian, "Energy of Cayley Graphs for Alternating Groups", Southeast Asian Bulletin of Mathematics, 44(6), 789-795, 2020.

3 Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin and Ahmad Erfanian, "The Energy of Cayley Graphs for Symmetric Groups of Order 24", ASM Sc. J., Vol. 13, 2020, 1-6.

4 Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin, Ahmad Erfanian. "The Energy of Cayley Graphs for a Generating Subset of the Dihedral Groups", MATEMATIKA: MJIAM, 35(3), 371–376, 2019.

5 Amira Fadina Ahmad Fadzil, Nor Haniza Sarmin and Rabiha Mahmoud Birkia, "The energy of conjugacy class graphs of some metabelian groups", AIP Conference Proceedings, 1974, 030014 (2018).

DR. NUR IDAYU ALIMON

1 Sanhan Muhammad Salih Khasraw, Nor Haniza Sarmin, Nur Idayu Alimon, Nabilah Najmuddin and Ghazali Semil @ Ismail. "Sombor Index and Sombor Polynomial of the Noncommuting Graph Associated to Some Finite Groups", Journal of Advanced Research in Applied Sciences and Engineering Technology, 42(2): 112-121, 2024.

2 Fariz Maulana, Muhammad Zulfikar Aditya, Erma Suwastika, Intan Muchtadi-Alamisyah, Nur Idayu Alimon, Nor Haniza Sarmin. "On The Topological Indices of Zero Divisor Graphs of Some Commutative Rings", Journal of Applied Mathematics and Informatics, 42(3): 663-680, 2024.

3 Nur Idayu Alimon, Nor Haniza Sarmin and Ahmad Erfanian. "On the Szeged Index and Its Noncommuting Graph", Jurnal Teknologi, 85(3), 105-110, 2023.

DR. NUR IDAYU ALIMON (contd.)

4 Nur Idayu Alimon, Nor Haniza Sarmin and Ahmad Erfanian. "On Topological Indices of a Graph Associated to the Direct Product of Two Groups", Kongzhi yu Juece/Control and Decision Journal, 38(03), 1173-1181, 2023.

5 Nor Haniza Sarmin, Nur Idayu Alimon and Ahmad Erfanian. "Topological Indices of the Non-commuting Graph for Generalised Quaternion Group", Bulletin of the Malaysian Mathematical Sciences Society, 43, 3361–3367, 2020.

6 Nur Idayu Alimon, Nor Haniza Sarmin and Ahmad Erfanian, "The Harary Index of the Non-commuting Graph for Dihedral Groups", Southeast Asian Bulletin of Mathematics, 44(6), 763-768, 2020.

7 Nur Idayu Alimon, Nor Haniza Sarmin, Ahmad Erfanian. "The Topological Indices of the Non-commuting Graph for Symmetric Groups", ASM Sc. J., Vol. 13, 2020, 1-5.

8 Nur Idayu Alimon, Nor Haniza Sarmin and Ahmad Erfanian, "The Szeged and Wiener Indices for Coprime Graph of Dihedral Groups", AIP Conference Proceedings 2266, 060006, 2020.

DR. NUR IDAYU ALIMON (contd.)

9 Nur Idayu Alimon, Nor Haniza Sarmin, Ahmad Erfanian. "The Wiener and Zagreb Indices of Conjugacy Class Graph of the Dihedral Groups", MATEMATIKA, 35(1), 51-57, 2019.

10 Nur Idayu Alimon, Nor Haniza Sarmin, Amira Fadina Ahmad Fadzil. "The Energy of Four Graphs of Some Metacyclic 2-Groups", Malaysian Journal of Fundamental and Applied Sciences, 14(1), 59-66, 2018.

GHAZALI SEMIL @ ISMAIL

1 Ghazali Semil @ Ismail, Nor Haniza Sarmin, Nur Idayu Alimon and Fariz Maulana. "General Zeroth-Order Randić Index of the Zero Divisor Graph for Some Commutative Rings", AIP Conf. Proc. 2905, 070005, 2024.

2 Ghazali Semil @ Ismail, Nor Haniza Sarmin, Nur Idayu Alimon and Fariz Maulana. "The First Zagreb Index of the Zero Divisor Graph for the Ring of Integers Modulo Power of Primes", Malaysian Journal of Fundamental and Applied Sciences, 19(5), 892-900, 2023.

3 Ghazali Semil @ Ismail, Nor Haniza Sarmin, Nur Idayu Alimon and Fariz Maulana. "General Zeroth-Order Randić Index of Zero Divisor Graph for the Ring of Integers Modulo p^n", AIP Conference Proceedings, 2975, 020002(1-6), 2023.

My Collaborators Around the World

United States
- Kappe, Luise-Charlotte
- Goode, Elizabeth
- Morse, Robert
- Beuerle, James R
- Visscher, Matthew P

Iran
- Davvaz, Bijan
- Darafsheh, Mohammad Reza
- Erfanian, Ahmad
- Molaei, Mohammad Reza
- Tolue, Behnaz
- Rashid, Samad
- Moradipour, Kayvan
- Karimi, Fariba A.Hediyeh
- Jahandideh, Mariam
- Barakat, Yasamin
- Ghouchan, M Farrokhi
- Jafarabadi, Hossein M

Iraq
- Khasraw, Sanhan
- Samin, Nizar Majeed
- Mohialdeen, Vian Salah
- Abdulla, Chenar

United Kingdom
- Subramaniam, K. G

Afghanistan
- Mudaber, Hassan

United Arab Emirates
- Turaev, Sherzod

Qatar
- El-Sanfaz, Mustafa Anis

Libya
- Omer, Sanaa Mohamed Saled
- Awad, Emtinan
- Saleh, Masaoud Omran
- Ramadan, Fatma

Saudi Arabia
- Birkia, Rabiha
- Al-Rehali, Suad
- Alshammari, Maryam

Italy
- Malinin, Dmitry A

Nigeria
- Gambo, Ibrahim

India
- Suji, Suresh
- A. Divya
- A. Manimaran

South Africa
- Engelbrecht, Andries Petrus
- Russo, Francesco G

Pakistan
- Khan, Asghar
- Khan, Faiz Muhammad
- Khan, Hidayatullah
- Ghafoor, Asad

South Korea
- Jun, Young Bae

Taiwan
- Gan, Yee Siang

Brunei Darussalam
- Venkat, Ibrahim

Indonesia
- Herwindiati, Dyah
- Muchtadi, Intan
- Wijayanti, Indah Emilia
- Wardhana, Adhitya
- Alfi Yusrotis Zakiyyah
- Siti Zahidah
- Maulana, Fariz
- Swastika, Erma
- Putra, Lalu Riski Wirendra

Acknowledgement

- This work was funded by Ministry of Higher Education Malaysia (MoHE) under Fundamental Research Grant Scheme (FRGS/1/2020/STG06/UTM/01/2) and Universiti Teknologi Malaysia Fundamental Research Grant (UTMFR) Vote Number 20H70. Additionally, this research is supported by the ITB International Research Grant (contract number : 8693/IT1.B07.1/TA.00/2023).
- Special thanks to Dr. Rabiha Mahmoud Birkia, Dr. Amira Fadina Ahmad Fadzil, Dr. Nur Idayu Alimon, and Mr. Ghazali Semil @ Ismail for their contributions to this research.

THANK YOU

Email: nhs@utm.my
https://people.utm.my/nizasarmin/