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RESEARCH ARTICLE

The Minimum Degree Energy of the Cayley Graph Associated to the Dihedral Group of Order Six with Subsets of Order Two and Three

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Abstract The energy of a simple graph in graph theory is defined as the sum of the absolute values of the eigenvalues of the graph's adjacency matrix, a concept inspired by Hückel Molecular Orbital theory. Chemists originally used this idea to estimate the energy associated with π-electron orbitals in conjugated hydrocarbons. The minimum degree energy, on the other hand, is defined as the sum of the absolute values of the eigenvalues of the graph's minimum degree matrix. A Cayley graph associated to a finite group with a subset S is defined as a graph in which the vertices are the elements of the group and two vertices v_1 and v_2 are joined with an edge if and only if v_2 is equal to the product of s and v_1 for some elements s in the subset S. In this research, we compute the minimum degree energy of Cayley graphs associated with the dihedral group of order six, focusing on subsets of orders two and three. The process involves constructing the Cayley graph for each subset, determining the minimum degree matrix, and calculating the corresponding eigenvalues. The findings indicate that for subsets of order two, the minimum degree energy is 16, while for subsets of order three, the minimum degree energies is 18 or 24. Notably, the minimum degree energy is an even number for all cases.

Keywords: Cayley graph, minimum degree energy of graph, dihedral group, graph theory, group theory.

Introduction

The study of Cayley graphs, initiated by Arthur Cayley in 1878, has developed into a rich and fascinating area within algebraic graph theory, offering deep connections between group theory and graph theory. A Cayley graph is a visualization of the structure of a group, where each vertex represents a group element, and edges reflect group operations. Specifically, for a group G and a subset $S \subset G$, the Cayley graph $Cay(G, S)$ without the identity element is formed by connecting vertices v_1 and v_2 with an edge if there exists an element $s \in S$, such that the product of s and v_1 equals v_2 [1]. This structure is particularly useful in studying the properties of groups by analyzing the corresponding graphs, and over time, many aspects of Cayley graphs have been examined. For instance, Adiga and Ariamanesh have specifically studied the Cayley graphs on symmetric groups in 2012 [2]. Furthermore, Ramaswamy and Veena have determined the energy of unitary Cayley graphs [3] which was extended from Balakrishnan in 2004 [4]. Additionally, other writers expanded the research to include other varieties of Cayley graphs, including prime power Cayley graph [5], intersection power Cayley graph [6], and normal edge-transitive Cayley graph [7].

Cayley graphs, due to their strong algebraic structure, have significant applications in various fields, including computer science, chemistry, and communication networks. In computer science, Cayley graphs are used to design efficient network topologies for parallel processing and distributed computing systems. Their symmetrical properties ensure optimal routing and fault tolerance, making them ideal

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candidates for constructing robust interconnection networks. For example, the Cayley graph technique has been demonstrated by Cooperman *et al.* to handle rearrangement issues and the creation of interconnection networks for parallel CPUs [8]. In chemistry, the energy of Cayley graphs can model molecular structures, providing insights into the stability and reactivity of chemical compounds, particularly in studying conjugated systems using graph-theoretical models derived from molecular orbital theory [9].

Among the various spectral properties of graphs, graph energy has emerged as an important concept since its introduction by Gutman in 1978. [10]. The sum of the absolute values of the eigenvalues of a graph's adjacency matrix was used to describe its energy in this context. The collection of all the graph's eigenvalues is called its spectrum. Let R be a simple graph that is finite and also undirected, with vertex set $V(R)$ and edge set $E(R)$. The graph R has n vertices, and each vertex is designated with $v_1, v_2, \ldots, v_n$. If vertices v_i and v_j are adjacent, the adjacency matrix, $A(R)$, of graph R is a square matrix of dimension $n \times n$, with an (i,j)-entry of 1, and 0 otherwise [11].

Initially, graph energy was motivated by the Hückel Molecular Orbital Theory (HMO) in chemistry, which sought to understand the energy levels associated with π-electrons in conjugated hydrocarbons. The adjacency matrix captures the connections between vertices in a graph, and the spectrum of this matrix provides insight into the graph's structure. Over time, this concept has been applied beyond chemistry to various areas in mathematics, particularly in the study of Cayley graphs. For example, recent studies have explored the energy of Cayley graphs associated with non-abelian groups, such as the alternating groups, highlighting their unique spectral properties [12].

In 2010, Adiga and Swamy expanded the notion of graph energy by introducing the concept of minimum degree energy. Unlike traditional graph energy, which relies on the adjacency matrix, minimum degree energy is based on the minimum degree matrix of a graph. The minimum degree matrix is defined similarly to the adjacency matrix but incorporates the minimum degree of the vertices [13]. This concept has attracted attention because it offers a new perspective on graph energy, emphasizing the role of vertex degrees in determining the spectral properties of the graph. Since its introduction, researchers have explored the minimum degree energy of various graph classes, particularly those associated with groups. In the case of regular graphs, for example, Basavanagoud and Jakkannavar have calculated the lowest degree energy and obtained bounds for the greatest minimum degree eigenvalue and the minimum degree energy [14]. Additionally, the study of the minimum degree energy of a particular graph of groups has been extended by Rao [15] and Romdhini and Nawawi [16]. This study has practical implications where the results can be applied to optimize network designs in computer science or to model molecular structures in chemistry, where the energy of graphs plays a crucial role in predicting stability and reactivity.

In this article, we investigate the minimum degree energy of the Cayley graphs associated with the dihedral group of order six, D_6, using subsets of order two and three. The dihedral group, denoted as D_n, is a well-known group that characterizes the symmetry properties of a regular polygon with n sides. It consists of both rotational and reflectional symmetries, making it a rich subject for exploration in the context of Cayley graphs. The group of equilateral triangle symmetries with six elements, three rotations and three reflections, is known as the order six dihedral group, or D_6 [17]. In this research, two specific cases of subsets of D_6 are considered, which are subsets of order two and subsets of order three. These subsets correspond to different selections of elements from the group, which in turn affect the structure of the resulting Cayley graphs. For example, a subset of order two may consist of a rotation and a reflection, while a subset of order three could include two rotations and a reflection.

For each subset, we construct the corresponding Cayley graph and compute its minimum degree matrix. The eigenvalues of the matrix are then calculated to determine the minimum degree energy. By comparing the minimum degree energies of Cayley graphs with different subsets, we gain insight into how the choice of subset influences the spectral properties of the graph.

Materials and Methods

The research began by constructing Cayley graphs associated with the dihedral groups of order six using subsets of order two and three, applying the formal definition of the Cayley graph. After constructing these graphs, the next step involved obtaining their minimum degree matrices using the definition of the minimum degree matrix. Once the matrices were defined, their eigenvalues were computed using the definition of the eigenvalues. Finally, to determine the minimum degree energy of each Cayley graph, we applied the definition of minimum degree energy, which states that it is the sum of the absolute values of all minimum degree eigenvalues of the graph. Figure 1 below illustrates the detailed research methodology for this research, outlining each step from graph construction to eigenvalue computation and energy determination.

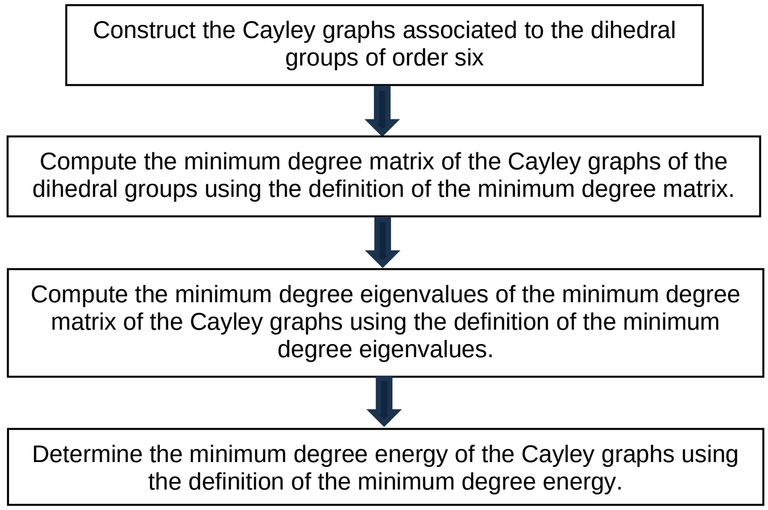


Figure 1. Research methodology flowchart

Preliminaries

In this section, important definitions used in the research are provided. These definitions come from graph theory, group theory, and linear algebra, which are needed to understand the research. This section introduces key terms such as dihedral groups, Cayley graphs, eigenvalues, and minimum degree matrix to provide a clear foundation for the study.

Definition 1 [17] Dihedral Groups
The dihedral group of order $2n$, denoted by D_{2n}, is the group of symmetries of an n-gon. These symmetries include rotations, denoted by R, and reflections, denoted L. The group presentation is $D_{2n} = \langle R, L|\ R^n = L^2 = 1 \text{ and } LRL = R^{-1}\rangle$.

Definition 2 [1] Cayley Graph of a Group
Let S be a subset of G such that $1 \notin S$ and that $S = S^{-1}$; in other words, $s \in S$ if and only if $s^{-1} \in S$. Let G be a finite group with identity 1. The following is the definition of the $Cay(G, S)$ on G with subset S:

- the elements of G are the vertices.
- For every s in S, there exists an edge between v_1 and v_2 if and only if $v_2 = sv_1$.

The set of edges is denoted as $E\left(Cay(G, S)\right) = \left\{\{v_{i,}v_{i,}\}|v_i \text{ is adjacent to } v_j\right\}$.
Remark: The relation between the two vertices can also be rewritten as $v_2v_1^{-1} = s$ for some $s \in S$.

Definition 3 [1] Complete Graph
Every vertex in a complete graph K_n is adjacent to every other vertex, and the graph has n vertices.

Definition 4 [18] Union of Graph
Let a graph G have two subgraphs, G_1 and G_2. The subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ is the union $G_1 \cup G_2$ of G_1 and G_2.

Note: The union of m copies of K_n, that is $\underbrace{K_n \cup K_n \cup \cdots \cup K_n}_{m \text{ copies}}$, is denoted by mK_n.

Definition 5 [18] Bipartite Graph and Complete Bipartite Graph
A graph that has its vertex set divided into two subsets, X and Y, so that each edge has one end in X and one end in Y, is said to be bipartite. This type of partition (X, Y) is known as a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) where every vertex of X is adjacent to every vertex of Y; such a graph is represented by $K_{m,n}$ if $|X| = m$ and $|Y| = n$.

Definition 6 [18] Cycle Graph
A cycle is a closed path with distinct internal vertices and an origin. A k-cycle, represented as C^k, is a cycle of length k.

Definition 7 [11] Characteristic Polynomial
Let A be a matrix of size $n \times n$. The determinant $\det(A - \lambda I)$, a polynomial in the (complex) variable λ of degree n, is the characteristic polynomial of A. The characteristic equation of an object is $\det(A - \lambda I) = 0$.

Definition 8 [11] Eigenvalues of a Matrix
The roots of the characteristic equation $\det(A - \lambda I) = 0$ of A are called the eigenvalues of A.

Definition 9 [13] Minimum Degree Matrix
Let R be a simple graph with n vertices $v_1, v_2, \dots, v_n$ and let $d_i = \deg(v_i)$ be the degree of v_i, $i = 1,2, \dots, n$. The minimum degree matrix of the graph R is defined by $M(R) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} \min\{d_i, d_j\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the minimum degree matrix $M(R)$ is defined by

$$g(R : \lambda) = \det(\lambda I - M(R)).$$

Definition 10 [13] Minimum Degree Energy
Let $M(R)$ be the minimum degree matrix of a graph R and $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues. Then, the minimum degree energy of the graph R is defined as

$$\varepsilon_M(R) = \sum_{i=1}^{n} |\lambda_i|.$$

Results and Discussion

This section presents the main results through a series of theorems. The Cayley graphs associated with the dihedral group D_6 are constructed for subsets S of orders two and three. After building these graphs, the minimum degree matrices are determined, and the minimum degree energy is calculated for each graph. These calculations help to investigate how the selection of subsets from the group affects the Cayley graphs' structure. Each result is clearly presented in the theorems, which provide a detailed understanding of the graph and their connection to the dihedral group.

The Cayley Graph Associated to the Dihedral Group of Order Six with Subsets of Order Two and Three

The Cayley graphs associated to the order six dihedral group, containing subsets S of order two and three, are constructed in this section. From Definition 1, $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. By Definition 2, the subsets of order two of D_6 are $\{R_{120}, R_{240}\}, \{L_1, L_2\}, \{L_1, L_3\}$ and $\{L_2, L_3\}$, while order three subsets of D_6 are $\{L_1, R_{120}, R_{240}\}, \{L_2, R_{120}, R_{240}\}, \{L_3, R_{120}, R_{240}\}$ and $\{L_1, L_2, L_3\}$. The results of Cayley graphs are illustrated in four theorems.

Theorem 1 Let D_6 be the order six dihedral group. Then the Cayley graph of D_6 with subsets of order two consisting of two rotations, $S = \{L_i, L_j\}$, is $Cay(D_6, S) = C_6$, for $i, j \in \{1,2,3\}, i \neq j$.

Proof Let D_6 be the order six dihedral group, defined as $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Consider $Cay(D_6, S)$ to be the Cayley graph of D_6 with subsets S of order three, namely $\{L_1, L_2\}, \{L_1, L_3\}$ and $\{L_2, L_3\}$. According to Definition 2, the vertex set of the Cayley graph, $V\left(Cay\left(D_6, \{L_i, L_j\}\right)\right) = D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Next, to determine the edges of the graph, the Cayley table of D_6 (Table 1) is used. Based on Definition 2, the vertices v_1 and v_2 are connected by an edge if and only if $v_2 = sv_1$ for $s \in \{L_i, L_j\}$, and v_1, v_2 in D_6, which can be written as $v_2 = L_1 v_1$. Thus, the edges are formed based on this multiplication rule in the group.

Table 1. The Cayley table of D_6

	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_0	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_{120}	R_{120}	R_{240}	R_0	L_3	L_1	L_2
R_{240}	R_{240}	R_0	R_{120}	L_2	L_3	L_1
L_1	L_1	L_2	L_3	R_0	R_{120}	R_{240}
L_2	L_2	L_3	L_1	R_{240}	R_0	R_{120}
L_3	L_3	L_1	L_2	R_{120}	R_{240}	R_0

The edge set of the Cayley graph, $E\left(Cay\left(D_6,\{L_i,L_j\}\right)\right)$ is then obtained. Hence, by Definition 2, the Cayley graph of D_6 with the subset $\{L_i,L_j\}$ can be drawn as in Figure 2.

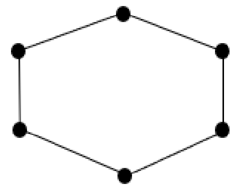


Figure 2. The Cayley graph of D_6 with the subset $S=\{L_i,L_j\}$.

Theorem 2 Let D_6 be the order six dihedral group and $Cay(D_6,\{R_{120},R_{240}\})$ be the Cayley graph of D_6 with the subset $\{R_{120},R_{240}\}$. Then, $Cay(D_6,\{R_{120},R_{240}\})=2K_3$, where $2K_3$ is the union of two complete graphs.

Proof Let D_6 be the order six dihedral group, $D_6=\langle R_0,R_{120},R_{240},L_1,L_2,L_3\rangle$. Consider $Cay(D_6,S)$ to be the Cayley graph of D_6 with subsets S of order two, namely $\{R_{120},R_{240}\}$. According to Definition 2, the vertex set of the Cayley graph, $V(Cay(D_6,\{R_{120},R_{240}\}))=D_6=\langle R_0,R_{120},R_{240},L_1,L_2,L_3\rangle$. Next, to determine the edges of the graph, the Cayley table of D_6 (Table 1) is used. Based on Definition 2 the vertices v_1 and v_2 are connected by an edge if and only if $v_2=sv_1$ for $s\in\{R_{120},R_{240}\}$, and v_1,v_2 in D_6, which can be written as $v_2=R_{120}v_1$. Thus, the edge set of the Cayley graph is given by $E\left(Cay(D_6,\{R_{120},R_{240}\})\right)=\{\{R_0,R_{120}\},\{R_{120},R_{240}\},\{R_{240},R_0\},\{L_1,L_3\},\ \{L_2,L_1\},\{L_3,L_2\}\}$. Hence, by Definition 2, the Cayley graph of D_6 with the subset $\{R_{120},R_{240}\}$ can be drawn as in Figure 3.

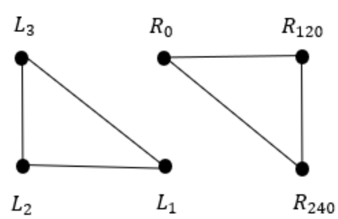
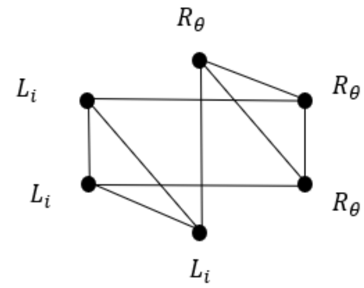


Figure 3. The Cayley graph of D_6 with the subset $\{R_{120},R_{240}\}$

Theorem 3 Let D_6 be the order six dihedral group. Then the Cayley graph of D_6 with subsets of order 3 consisting of one reflection and two rotations $S=\{L_i,R_{120},R_{240}\}$, $i\in\{1,2,3\}$, $Cay(D_6,S)$ is as shown in Figure 4.



where $i \in \{1,2,3\}$ and $\theta \in \{0, 120, 240\}$.

Figure 4. The Cayley graph of D_6 with the subset $\{L_i, R_{120}, R_{240}\}$

Proof Let D_6 be the order six dihedral group, $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Consider $Cay(D_6, S)$ to be the Cayley graph of D_6 with subsets S of order three, namely $\{L_1, R_{120}, R_{240}\}, \{L_2, R_{120}, R_{240}\}$ and $\{L_3, R_{120}, R_{240}\}$. According to Definition 2, the vertex set of the Cayley graph, $V\big(Cay(D_6, \{L_i, R_{120}, R_{240}\})\big) = D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Next, to determine the edges of the graph, the Cayley table of D_6 (Table 1) is used. Based on Definition 2, the vertices v_1 and v_2 are connected by an edge if and only if $v_2 = sv_1$ for $s \in \{L_i, R_{120}, R_{240}\}$, and v_1, v_2 in D_6, which can be written as $v_2 = L_1 v_1$. The edge set of the Cayley graph $E\big(Cay(D_6, \{L_i, R_{120}, R_{240}\})\big)$ is then obtained. Hence, by Definition 2, the Cayley graph of D_6 with the subset $\{L_i, R_{120}, R_{240}\}$ can be drawn as in Figure 4.

Theorem 4 Let D_6 be the order six dihedral group and $Cay(D_6, \{L_1, L_2, L_3\})$ be the Cayley graph of D_6 with the subset $\{L_1, L_2, L_3\}$. Then, $Cay(D_6, \{L_1, L_2, L_3\}) = K_{3,3}$, where $K_{3,3}$ is the complete bipartite graph.

Proof Let D_6 be the order six dihedral group, $D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Consider $Cay(D_6, S)$ to be the Cayley graph of D_6 with subsets S of order three, namely $\{L_1, L_2, L_3\}$. According to Definition 2, the vertex set of the Cayley graph, $V\big(Cay(D_6, \{L_i, R_{120}, R_{240}\})\big) = D_6 = \langle R_0, R_{120}, R_{240}, L_1, L_2, L_3 \rangle$. Next, to determine the edges of the graph, the Cayley table of D_6 (Table 1) is used. Based on Definition 2, the vertices v_1 and v_2 are connected by an edge if and only if $v_2 = sv_1$ for $s \in \{L_1, L_2, L_3\}$, and v_1, v_2 in D_6, which can be written as $v_2 = L_1 v_1$. Thus, the edge set of the Cayley graph is given by $E\big(Cay(D_6, \{L_1, L_2, L_3\})\big) = \{\{L_1, R_0\}, \{L_2, R_{120}\}, \{L_3, R_{240}\}, \{L_2, R_0\}, \{L_3, R_{120}\}, \{L_1, R_{240}\}, \{L_3, R_0\}, \{L_1, R_{120}\}, \{L_2, R_{240}\}\}$. Hence, by Definition 2, the Cayley graph of D_6 with the subset $\{L_1, L_2, L_3\}$ can be drawn as in Figure 5.

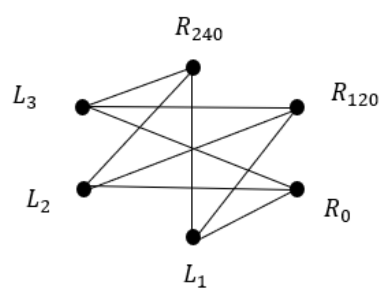


Figure 5. The Cayley graph of D_6 with the subset $\{L_1, L_2, L_3\}$.

The Minimum Degree Energy of the Cayley Graph Associated to the Dihedral Group of Order Six with Subsets of Order Two and Three

In this section, the minimum degree energy of the Cayley graph associated with the order six dihedral group is computed and analyzed. The Cayley graphs, which were constructed in the previous section using subsets of order two and three from the dihedral group, serve as the foundation for these calculations.

Theorem 5 Let D_6 be the order six dihedral group. Then, the minimum degree energy of the Cayley graph of D_6 with the subset S of order two, is $\varepsilon_M\big(Cay(D_6, S)\big) = 16$.

Proof Let R be a graph, D_6 be the order six dihedral group and $Cay(D_6, S)$ be the Cayley graph of D_6 with the subset S of order two. First, let $S = \{R_{120}, R_{240}\}$. Based on Definition 8 of the minimum degree matrix, $V(R)$ is used to index the rows and columns of $M(R)$, namely $v_1, v_2, v_3, v_4, v_5, v_6$ where $v_1 = R_0, v_2 = R_{120}, v_3 = R_{240}, v_4 = L_1, v_5 = L_2, v_6 = L_3$. Since $Cay(D_6, \{R_{120}, R_{240}\}) = 2K_3$, the corresponding minimum degrees of all vertices have the entry 2, otherwise 0. This is because the minimum degrees of all vertices of $Cay(D_6, \{R_{120}, R_{240}\})$ are 2. Thus, the minimum degree matrix of $Cay(D_6, \{R_{120}, R_{240}\})$ is obtained as follows:

$$M(R) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{bmatrix} \end{array}.$$

Then, the characteristic polynomial of $M(R)$ is $g(M(R), \lambda I) = \det(M(R) - \lambda I)$.

$$\det(M(R) - \lambda I) = \begin{vmatrix} -\lambda & 2 & 2 & 0 & 0 & 0 \\ 2 & -\lambda & 2 & 0 & 0 & 0 \\ 2 & 2 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 2 & 2 \\ 0 & 0 & 0 & 2 & -\lambda & 2 \\ 0 & 0 & 0 & 2 & 2 & -\lambda \end{vmatrix} = (-\lambda + 4)^2 (\lambda - 2)^4.$$

By Definition 7, the eigenvalues are $\lambda_1 = 4$ with multiplicity 2 and $\lambda_2 = 2$ with multiplicity 4. According to Definition 9, the minimum degree energy of $Cay(D_6, \{R_{120}, R_{240}\})$ is given by $\varepsilon_M\big(Cay(D_6, \{R_{120}, R_{240}\})\big) = 2|4| + 4|2| = 16$. The proof for $\varepsilon_M\big(Cay(D_6, \{L_1, L_2\})\big), \varepsilon_M\big(Cay(D_6, \{L_1, L_3\})\big)$ and $\varepsilon_M\big(Cay(D_6, \{L_2, L_3\})\big)$ follow a similar approach to that of $\varepsilon_M\big(Cay(D_6, \{R_{120}, R_{240}\})\big)$. Hence, the energy of the Cayley graph of D_6 with subset S of order two is given by $\varepsilon\big(Cay(D_6, S)\big) = 16$.

Theorem 6 Let D_6 be the order six dihedral group. Then, the minimum degree energy of the Cayley graph of D_6 with the subset S of order three for $\{L_1, R_{120}, R_{240}\}$, $\{L_2, R_{120}, R_{240}\}$ and $\{L_3, R_{120}, R_{240}\}$, is $\varepsilon_M\big(Cay(D_6, S)\big) = 24$.

Proof Let R be a graph, D_6 be the order six dihedral group and $Cay(D_6, S)$ be the Cayley graph of D_6 with the subset S of order three. First, let $S = \{L_1, R_{120}, R_{240}\}$. Based on Definition 8 of the minimum degree matrix, $V(R)$ is used to index the rows and columns of $M(R)$, namely $v_1, v_2, v_3, v_4, v_5, v_6$ where $v_1 = R_0, v_2 = R_{120}, v_3 = R_{240}, v_4 = L_1, v_5 = L_2, v_6 = L_3$. The corresponding minimum degrees of all vertices have the entry 3, otherwise 0. This is because the minimum degrees of all vertices of $Cay(D_6, \{L_1, R_{120}, R_{240}\})$ are 3. Thus, the minimum degree matrix of $Cay(D_6, \{L_1, R_{120}, R_{240}\})$ is obtained as follows:

$$M(R) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 3 & 3 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 3 & 3 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 3 & 3 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 0 & 0 & 3 & 3 & 3 & 0 \end{bmatrix} \end{array}.$$

Then, the characteristic polynomial of $M(R)$ is $g(M(R), \lambda I) = \det(M(R) - \lambda I)$.

$$\det(M(R) - \lambda I) = \begin{vmatrix} -\lambda & 3 & 3 & 3 & 0 & 0 \\ 3 & -\lambda & 3 & 0 & 3 & 0 \\ 3 & 3 & -\lambda & 0 & 0 & 3 \\ 3 & 0 & 0 & -\lambda & 3 & 3 \\ 0 & 3 & 0 & 3 & -\lambda & 3 \\ 0 & 0 & 3 & 3 & 3 & -\lambda \end{vmatrix} = \lambda^2 (\lambda - 3)(\lambda + 6)^2 (\lambda - 9).$$

By Definition 7, the eigenvalues are $\lambda_1 = 0$ with multiplicity 2, $\lambda_2 = 3$ with multiplicity 1, $\lambda_3 = -6$ with multiplicity 2 and $\lambda_4 = 9$ with multiplicity 1. According to Definition 9, the minimum degree energy of $Cay(D_6, \{L_1, R_{120}, R_{240}\})$ is given by $\varepsilon_M\big(Cay(D_6, \{L_1, R_{120}, R_{240}\})\big) = 2|0| + |3| + 2|-6| + |9| = 24$. The proof for $\varepsilon_M\big(Cay(D_6, \{L_2, R_{120}, R_{240}\})\big)$and $\varepsilon_M\big(Cay(D_6, \{L_3, R_{120}, R_{240}\})\big)$ follow a similar approach to that of $\varepsilon_M\big(Cay(D_6,\{L_1,R_{120},R_{240}\})\big)$. Hence, the energy of the Cayley graph of D_6 with subset S of order three is given by $\varepsilon_M\big(Cay(D_6,S)\big)=24$. □

Theorem 7 Let D_6 be the order six dihedral group and $Cay(D_6,\{L_1,L_2,L_3\})$ be the Cayley graph of D_6 with the subset $\{L_1,L_2,L_3\}$. Then, the minimum degree energy of $Cay(D_6,\{L_1,L_2,L_3\})$, is $\varepsilon_M\big(Cay(D_6,\{L_1,L_2,L_3\})\big)=18$.

Proof Let R be a graph, D_6 be the order six dihedral group and $Cay(D_6,S)$ be the Cayley graph of D_6 with the subset $S=\{L_1,L_2,L_3\}$. Based on Definition 8 of the minimum degree matrix, $V(R)$ is used to index the rows and columns of $M(R)$, namely v_1,v_2,v_3,v_4,v_5,v_6 where $v_1=R_0, v_2=R_{120}, v_3=R_{240}, v_4=L_1, v_5=L_2, v_6=L_3$. The corresponding minimum degrees of all vertices have the entry 3, otherwise 0. This is because the minimum degrees of all vertices of $Cay(D_6,\{L_1,L_2,L_3\})$ are 3. Thus, the minimum degree matrix of $Cay(D_6,\{L_1,L_2,L_3\})$ is obtained as follows:

$$M(R)=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6 \end{array}\begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array}\\ \begin{bmatrix} 0&0&0&3&3&3\\ 0&0&0&3&3&3\\ 0&0&0&3&3&3\\ 3&3&3&0&0&0\\ 3&3&3&0&0&0\\ 3&3&3&0&0&0 \end{bmatrix} \end{array}.$$

Then, the characteristic polynomial of $M(R)$ is $g(M(R),\lambda I)=\det(M(R)-\lambda I)$.

$$\det(M(R)-\lambda I)=\begin{vmatrix} -\lambda&0&0&3&3&3\\ 0&-\lambda&0&3&3&3\\ 0&0&-\lambda&3&3&3\\ 3&3&3&-\lambda&0&0\\ 3&3&3&0&-\lambda&0\\ 3&3&3&0&0&-\lambda \end{vmatrix}=\lambda^4(\lambda-9)(\lambda+9).$$

By Definition 7, the eigenvalues are $\lambda_1=0$ with multiplicity 4, $\lambda_2=9$ with multiplicity 1 and $\lambda_3=-9$ with multiplicity 1. According to Definition 9, the minimum degree energy of $Cay(D_6,\{L_1,L_2,L_3\})$ is given by $\varepsilon_M\big(Cay(D_6,\{L_1,L_2,L_3\})\big)=4|0|+|9|+|-9|=18$.

Conclusions

In conclusion, this study reveals clear patterns in the minimum degree energy of Cayley graphs associated with the dihedral group of order six. For subsets of order two, the minimum degree energy is consistently 16. In the other hand, for subsets of order three, the minimum degree energy varies, giving values of 24 or 18. One interesting finding is that, in all cases, the minimum degree energy is always an even number. These findings contribute to a deeper understanding of how the choice of subset affects the spectral properties of Cayley graphs, especially their minimum degree energy. Future research could look into whether similar patterns appear in Cayley graphs constructed from other types of groups, such as cyclic groups or symmetric groups. Additionally, exploring these patterns in different types of subsets of dihedral groups could help extend the research. This could help us better understand how the structure of groups relates to the energy of their graphs, giving a wider view on this connection.

Conflicts of Interest

The author(s) herein state that they have no conflict of interests with respect to the publishing of this work.

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