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On the uniqueness of almost prime submodules within cyclic uniserial modules

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Abstract

A uniserial module is a module that satisfies both ascending chain condition and descending chain condition, which makes a uniserial module an Artinian module and a Noetherian module at the same time. Recently an algebraic structure from ring theory, called almost prime ideal, is generalized into a module theory and called an almost prime submodule. Some researchers have examined the characterizations of this new algebraic structure in various types of modules. In this article, we provide some insights into the almost prime submodule of a uniserial cyclic module In this study, we have discovered that the non-zero almost prime submodule of the cyclic uniserial module is unique.

Subject Classification: *13C05, 13C10, 13C12.*

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1. Introduction

The almost prime submodule was generalized as a prime submodule which is an abstraction of the prime ideal ring theory. These almost prime submodule terms are given by Khashan which was a generalization of the almost prime ideal introduced by Anderson and Bataineh [2], Khashan gave some properties of the almost prime submodule to the multiplication module [6]. Wardhana et. all studies the almost prime submodule to the free module and finds that the almost prime submodule always be a prime submodule whenever the dimension of the submodule is less than the dimension of the module [8]. Later Juliana et. all found that the almost prime submodule must be prime in every case [5], the same thing occured to the CSM module [9]. There are many generalizations of prime submodules such as n-almost prime submodules, and for n is one, then we have an almost prime submodule [4]. Another study on finitely generated submodules explores the distinction between almost prime and prime submodules. While these submodules are typically equivalent, the research highlights specific cases where they differ, focusing on the decomposition of finitely generated modules over principal ideal domains for deeper insights [10].

Several other studies related to the generalization of prime submodules include radical semiprime submodules, almost small semiprime submodules, and J-prime submodules. The concept of radical semiprime submodules is introduced, where elements in the radical of a module satisfy certain conditions, implying membership in the submodule, enriching the understanding of prime submodules in module theory [1]. A further generalization is presented through the concept of almost semiprime submodules, where conditions on small semiprime submodules are extended and applied to duplication R-modules, providing new structural insights [7]. Finally, the notion of J-prime submodules is developed by incorporating the Jacobson radical to explore conditions under which module elements belong to the submodule or satisfy homomorphic properties. This introduces new characterizations of prime submodules, adding to the body of knowledge in commutative algebra and module theory [3]. Together, these works expand foundational theories and offer broader applicability in algebraic structures.

In this article, we discussed the almost prime submodule to the uniserial cyclic module over a principal ideal domain. The main result is that every cyclic uniserial module over has a unique non-zero almost prime submodule whenever the ring is a principal ideal domain.

2. Main results

The uniserial module was an obvious generalization of chainrings, which is its collections of all its submodules are totally ordered set, by inclusion [11]. Here we provide some basic terminologies.

Definition 2.1: Let V be an R-module, the chain submodules

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k = M$$

is called a sequence of V, and the factor of the sequence V are $\frac{V_{i+1}}{V_i}$ with $i = 0,1,2\ldots,k-1$

Two sequences of a module that had an equal number of factors are said to be equivalent if there exist some factors that are isomorphs. Let $\mathbb{Z}_{72}$ be a module over itself, this module had two sequences $\langle 0 \rangle \subset \langle 36 \rangle \subset \langle 6 \rangle \subset \mathbb{Z}_{72}$ and $\langle 0 \rangle \subset \langle 36 \rangle \subset \langle 9 \rangle \subset \mathbb{Z}_{72}$ that is equivalent.

Look at the example of the sequence of $\mathbb{Z}_{72}$ again, we can be embedded some submodules into the sequence to make the sequence longer, in this case, we have $\langle 0 \rangle \subset \langle 36 \rangle \subset \langle 18 \rangle \subset \langle 9 \rangle \subset \langle 3 \rangle \subset \mathbb{Z}_{72}$. This process is called sequence smoothing.

Definition 2.2: Let V be an R-module, the chain submodules

$$\{0\} = W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n = V$$

is called sequence smoothing of

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k = V$$

if W_j contained V_i for all $i \leq j$.

If we recalled the sequence $\langle 0 \rangle \subset \langle 36 \rangle \subset \langle 18 \rangle \subset \langle 9 \rangle \subset \langle 3 \rangle \subset \mathbb{Z}_{72}$, we can not make this sequence longer, we call this sequence a composition series.

Definition 2.3: Let V be an R-module, the chain submodules

$$\{0\} = W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n = V$$

is called a composition series if we can not apply sequence smoothing to this sequence.

Let W_1, W_2 be submodule such that $W_1 \subset W_2$, if there is no trivial submodule between W_1 and W_2 then we have W_2 / W_1 is a simple module.

Theorem 2.4: *Let V be an R-module, the chain submodules*

$$\{0\} = W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n = V$$

is a composition series if and only if $\frac{W_{i+1}}{W_i}$ *is a simple module for all* $i = 0, 1, 2, \ldots, n-1$.

Proof: See [11]

Theorem 2.5: (Jordan-Holder) *Let V be an R-module, two composition series must be equivalent.*

Proof: See [11]

By Theorem 2.5 we can conclude that, if a module had a composition series, then sequence smoothing can apply to any sequence to get a composition series. Now we can define the uniserial module.

Definition 2.6: Let V be an R-module, the module V is called a uniserial module if V has a unique composition series.

The module $\mathbb{Z}_4$ over integer is an example of a uniserial module, since it has $\langle 0\rangle \subset \langle 2\rangle \subset \mathbb{Z}_4$ as the unique composition series. We can see from the previous example that $\mathbb{Z}_{72}$ is not a uniserial module as it has two composition series $\langle 0\rangle \subset \langle 36\rangle \subset \langle 18\rangle \subset \langle 9\rangle \subset \langle 3\rangle \subset \mathbb{Z}_{72}$ and $\langle 0\rangle \subset \langle 24\rangle \subset \langle 8\rangle\ \langle 4\rangle \subset \langle 2\rangle \subset \mathbb{Z}_{72}$. The integer module is another example that is not a uniserial module since it does have not a composition series.

A prime submodule is the abstraction of the prime ideal that was introduced by Daun in 1978, later this term was generalized by Khashan in 2012. The challenge of this abstraction is to replace the multiplication in the ring since we have no multiplication in a module. Daun smartly introduced a term called fraction submodule to fill the gap [8].

Definition 2.7: Let V be an R-module, and W be its submodule. The fraction submodule of W is $(W:V) = \{r \in R \mid rV \subseteq W\}$.

It is clear that the fraction submodule is an ideal of the ring. With the fraction submodule, now we can define the prime submodule and the almost prime submodule.

Definition 2.8: Let V be an R-module. Proper submodule W is called a prime submodule if for every $v \in V$ and every $r \in R$ such that $rv \in W$ implies that $r \in (W:V)$ or $v \in W$.

Definition 2.9: Let V be an R-module. Proper submodule W is called a almost prime submodule if for every $v \in V$ and every $r \in R$ such that $rv \in W - (W:V)W$ implies that $r \in (W:V)$ or $v \in W$.

A prime submodule always an almost prime submodule by definition. But in $\mathbb{Z}$-module $\mathbb{Z}_{24}$, the submodule $\langle 8 \rangle$ is an almost prime submodule that is not a prime submodule.

To invetigate the prime submodule and the almost prime submodule, first we need to know its fraction submodule. For the integer modulo module over an integer, we have this formula.

Theorem 2.10: *Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module, where $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, with p_i are distinct prime numbers and $k_i \in \mathbb{N}$. If $W = \langle x \rangle$ is a proper submodule of $\mathbb{Z}_n$, then $(W:\mathbb{Z}_n) = \langle x \rangle$.*

Proof: Noted that if we assume that $(x,n) = 1$ then we have $1 = px + qn$ for some $p,q \in \mathbb{Z}$, hence $\overline{1} = p\overline{x}$. Then we have $\overline{1} \in W$, and $W = \mathbb{Z}_n$ which contradicts W proper, since $x < n$ we have $x \,|\, n$. Now let $W = \langle x \rangle$, and $(W:\mathbb{Z}_n) = \langle y \rangle$, it is clear that $x \in (W:\mathbb{Z}_n)$ then we need only to show that $(W:\mathbb{Z}_n) \subseteq x$. Since $y\mathbb{Z}_n \subseteq W = \langle x \rangle$ then $y\overline{1} = \overline{y} \in x$, hence $\overline{y} = k\overline{x}$. Then $cn = y - kx$, since $x \,|\, n$ then we have $x \,|\, y$, so we have $(W:\mathbb{Z}_n) \subseteq \langle x \rangle$, hence $(W:\mathbb{Z}_n) = \langle x \rangle$.

With the formula of the fraction submodule from Theorem 2.10, we have these formulas to characterize the prime submodule and the almost prime submodule of the integer modulo module over an integer.

Theorem 2.11: *Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module, where $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, , with p_i are distinct prime numbers and $k_i \in \mathbb{N}$. Let W be a non-trivial submodule of $\mathbb{Z}_n$, then*

1. *Submodule W is a prime submodule if and only if $W = \langle p_i \rangle$ for $i \in \{1,2,..m\}$.*
2. *Submodule W is an almost prime submodule if and only if $W = p_i^{l_i}$ for $l_i = 1$ or $l_i = k_i$.*

Proof: Note that since there exists a non-trivial submodule, then n is not a prime number.

1. Let W be a prime submodule, since N is a non-trivial submodule, then $W = \langle p_1^{l_1}\ p_2^{l_2} \dots p_m^{l_m} \rangle$ for $0 \le l_i \le k_i$ and $(W:\mathbb{Z}_n) = \langle p_1^{l_1}\ p_2^{l_2} \dots p_m^{l_m} \rangle$ based on Theorem 2.10. If there is $a,b \in \{1,2,..,m\}$, such that l_a and l_b are non-zero, then choose $r = p_1^{l_1}\ p_2^{l_2} \dots p_m^{l_m} \,/\, p_b^{l_a} \notin (W:\mathbb{Z}_n)$ and $v = \langle p_b^{l_b} \rangle \notin (W:\mathbb{Z}_n)$, so we have $rv = p_1^{l_1} p_2^{l_2} \dots p_m^{l_m} \in N$ which contradicted W is a prime submodule. Hence $W = \rangle p_i^{l_i} \rangle$ for

$i \in \{1,2,..,m\}$, but i can not excessed 1 since, if $i > 1$ then again we can choose $r = p_i \notin (W : M)$ and $v = p_i^{l_i - 1} \notin W$ where $rv = p_i^{l_i} \in W$ which contradicted W a prime submodule. Hence $W = \langle p_i \rangle$ for $i \in \{1,2,..,m\}$. Now let $W = \langle p_i \rangle$ for $i \in \{1,2,..,m\}$. If $r \in \mathbb{Z}$ and $v \in \mathbb{Z}_n$ such that $rv \in \langle p_i \rangle$, then $rv = kp_i$ for some $k \in \mathbb{Z}$. Hence $rv = kp_i + cn$ and we can conclude that $p_i \mid rm$ since $p_i \mid n$. But p_i is a prime number, then we have $p_i \mid r$ or $p_i \mid v$, hence $r \in (W : \mathbb{Z}_n)$ or $v \in \langle p_i \rangle$. Then by definition W is a prime submodule.

2. For the second part, if $W = \langle p_i^{l_i} \rangle$ for $l_i = 1$ then W is a prime submodule according to the first part of this proof, hence W is almost prime. Now let $l_i = k_i$, from Theorem 2.10 we have $(W : \mathbb{Z}_n) = \langle p_i^{k_i} \rangle$. Since $(p_1^{k_i} \ldots p_{i-1}^{k(i-1)}\, p_{i+1}^{k(i+1)} \ldots p_m^{k_m},\, p_i^{k_i}) = 1$, hence there exist $x, y \in \mathbb{Z}$ such that $1 = xp_1^{k_i} \ldots p_{i-1}^{k(i-1)}\, p_{i+1}^{k(i+1)} \ldots p_m^{k_m} + yp_i^{k_i}$, so we have $p_i^{k_i} = xn + yp_i^{k_i} p_i^{k_i}$. Then we have $\langle p_i^{k_i} \rangle = \langle p_i^{2k_i} \rangle$, hence $W - (W : \mathbb{Z}_n)$ $W = \langle p_i^{k_i} \rangle - \langle p_i^{k_i} \rangle \langle p_i k_i \rangle = \langle p_i^{k_i} \rangle - \langle p_i^{2k_i} \rangle = \varnothing$. We can conclude that W is almost prime. Now let W be an almost prime submodule, since W is a non-trivial submodule, then $W = \langle p_1^{l_1}\, p_2^{l_2} \ldots p_m^{l_m} \rangle$ for $0 \le l_i \le k_i$ and $(W : \mathbb{Z}_n) = \langle p_1^{l_1}\, p_2^{l_2} \ldots p_m^{l_m} \rangle$ based on Theorem 2.10, we have $(W : \mathbb{Z}_n)\, W = \angle p_1^{2l_1}\, p_2^{2l_2} \ldots p_m^{2l_m} \rangle$. If there is $a, b \in \{1,2,..,m\}$, such that l_a and l_b are non-zero, since W is not trivial, then without loss of generality we can assume that $0 < l_a$ and $l_b < k_b$. Choose $r = p_1^{l_1}\, p_2^{l_2} \ldots p_m^{l_m} \,/\, p_b^{l_a} \notin (W : \mathbb{Z}_n)$ and $v = \langle p_b^{l_b} \rangle \notin (W : \mathbb{Z}_n)$, so we have $rv = p_1^{l_1}\, p_2^{l_2} \ldots p_m^{l_m} \in N$, and $rv \notin (W : \mathbb{Z}_n)\, W$ since $l_b < 2l_b$, which contradicted W is an almost prime submodule. Hence $W = \langle p_i^{l_i} \rangle$ for $i \in \{1,2,\ \ldots,\ m\}$. If $l_i = 1$ then according to the first part, we have W as a prime submodule, hence almost prime. Now let's assume that $1 < l_i < k_i$, then we can choose $r = p_i \in (W : \mathbb{Z}_n) = \langle p_i^{l_i} \rangle$ and $v = p_i^{l_i - 1}$ such that $rv = p_i^{l_i} \in W$ and $rv = p_i^{l_i} \notin \langle p_i^{2l_i} \rangle = (W : \mathbb{Z}_n)W$, hence it contradicted to W almost prime, then $l_i = k_i$. So we can conclude $W = \langle p_i^{l_i} \rangle$ for $l_i = 1$ or $l_i = k_i$.

In the next theorem, we will show the requirement of an integer modulo module over an integer will be a uniserial module.

Theorem 2.12: *Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module. The module $\mathbb{Z}_n$ is a uniserial module if and only if n is a prime power.*

Proof: Let $\mathbb{Z}_n$ be a uniserial module. If n is not a prime power, then there are two different prime number p_1 and p_2 that divide n. Which is impossible, since we will have two sequences $\{0\} \subseteq \langle p_1 \rangle \subseteq \mathbb{Z}_n$ and $\{0\} \subseteq \langle p_2 \rangle \subseteq \mathbb{Z}_n$ that can be two different composition series, since $\mathbb{Z}_n$ is finite. Then n must be a prime power. Now let n be a prime power, then $n = p^k$ for p a prime number and $k \in \mathbb{N}$. Then clearly we have a unique decomposition series $\{0\} \subseteq \langle p^{k-1} \rangle \subseteq \ldots \subseteq \langle p^2 \rangle \subseteq \langle p \rangle \subseteq \mathbb{Z}_n$, hence $\mathbb{Z}_n$ is a uniserial module.

Based on Theorem 2.12, the integer modulo module consists of only one non-zero almost prime submodule.

Theorem 2.13: *Let $\mathbb{Z}_n$ be a $\mathbb{Z}$-module, and N be its submodule. If $\mathbb{Z}_n$ is a uniserial module, then $\mathbb{Z}_n$ has a unique non-zero almost prime submodule.*

Proof: Let $\mathbb{Z}_n$ be a uniserial module. Based on Theorem 5, then n is a prime power of a prime number p. Then according to Theorem 2.11, we have $\langle p \rangle$ is the only non-zero almost prime submodule of $\mathbb{Z}_n$.

Then we have these propositions.

Proposition 2.14: Let $\mathbb{Z}_n$ be a uniserial $\mathbb{Z}$-module with $n = p^k$ for p a prime number and $k \in \mathbb{N}$, then we have these properties

1. The decomposition series of $\mathbb{Z}_n$: $\{0\} \subseteq \langle p^{k-1} \rangle \subseteq \ldots \subseteq \langle p^2 \rangle \subseteq \langle p \rangle \subseteq \mathbb{Z}_n$.
2. The unique non-zero almost prime submodule of $\mathbb{Z}_n : \langle p \rangle$.
3. Zero submodules are the only almost prime submodule that is not prime submodule.

Proof: It is a direct result of Theorem 2.12 and Theorem 2.13 and the fact that $\langle p \rangle$ is also a prime submodule.

And for a more general module, we have a similar result.

Theorem 2.15: *Let V be a R-module, and W be its submodule. If V is a cyclic uniserial module, then V has a unique non-zero almost prime submodule.*

Proof: Since V is a uniserial module, then V has a composition series $\{0\} \subset V_1 \subset V_2 \subset \ldots V_n \subset V$, we will show you that V_n is the unique almost prime submodule of V. If $(V_n : V)\ V_n = V_n$, then we have $V_n - (V_m : V) = \varnothing$, hence by definition V_n is an almost prime submodule. Now we will show you that $(V_n : V)V_n = V_n$, since V is a cyclic module over a PID R, then we can assume that $(V_m : V) = \langle r \rangle$ for $r \in R$ and $V_m = \langle x \rangle = Rx$ for some $x \in V_m$. Now let $y \in (V_n : V)V_m$, then there exist $r_1 r \in \langle r \rangle$ and $sx \in \langle x \rangle$

such that $y = r_1 rsx$ for $r_1, s \in R$, hence $y \in \langle x \rangle = V_n$. Then we can conclude that V_m is almost prime.

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