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ON IDEAL BASED WEAKLY ZERO DIVISOR GRAPH AND ITS PROPERTIES

(*Graf Pembahagi Sifar Lemah Berasaskan Ideal dan Sifat-sifatnya*)

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ABSTRACT

For a commutative ring S having J as an ideal, the weakly zero divisor graph of S, denoted by $W\Gamma(S)$, has the zero divisors of S as its vertices, and two distinct vertices a and b are adjacent if there exist $r, s \in S\backslash\{0\}$ such that ar, bs and rs are all zero. An ideal based weakly zero divisor graph of S, denoted by $W\Gamma_J(S)$, is the graph with the vertex set $\{a \in S\backslash J : ab \in J$ for some $b \in S\backslash J\}$ and the edge set $\{(a, b) : ar \in J, bs \in J$ and $rs \in J$ for some $r, s \in S\backslash J\}$. The graph $W\Gamma_J(S)$ contains the ideal based zero divisor graph $\Gamma_J(S)$ as a subgraph and is identical to $W\Gamma(S)$ if $J = \{0\}$. In this article, the clique number and the connectivity of the graph $W\Gamma_J(S)$ are determined and the bounds on these numbers are provided. The clique number and connectivity of the graph $W\Gamma_J(S)$ are related to the clique number and connectivity of the graph $W\Gamma(S/J)$.

Keywords: zero divisor graph; connectivity; clique number; commutative ring

ABSTRAK

Bagi satu gelanggang komutatif S yang mempunyai J sebagai ideal, graf pembahagi sifar lemah bagi S, yang dilambangkan sebagai $W\Gamma(S)$, terdiri daripada pembahagi sifar dalam S sebagai set bucu, dan dua bucu berbeza a dan b adalah bersambung sekiranya terdapat $r, s \in S\backslash\{0\}$ dengan syarat ar, bs dan rs semuanya adalah sifar. Graf pembahagi sifar lemah berasaskan ideal, yang dilambangkan sebagai $W\Gamma_J(S)$, ialah graf dengan set bucu $\{a \in S\backslash J : ab \in J$ bagi beberapa $b \in S\backslash J\}$ yang mana dua bucu berbeza a dan b adalah bersambung jika wujud $r, s \in S\backslash J$ dengan syarat ar, bs dan rs semuanya di dalam J. Graf $W\Gamma_J(S)$ mengandungi graf pembahagi sifar berasaskan ideal $\Gamma_J(S)$ sebagai subgraf dan menjadi sama dengan $W\Gamma(S)$ apabila $J = \{0\}$. Dalam artikel ini, bilangan klik dan kebersambungan graf $W\Gamma_J(S)$ telah ditentukan, dan sempadan bagi nilai-nilai ini turut diberikan. Nilai bilangan klik dan kebersambungan bagi graf $W\Gamma_J(S)$ adalah berkaitan dengan bilangan klik dan kebersambungan graf $W\Gamma(S/J)$.

Kata kunci: graf pembahagi sifar; kebersambungan; bilangan klik; gelanggang komutatif

1. Introduction

The zero divisor graph of a commutative ring S is the graph $\Gamma(S)$ with vertices $Z(S)^* = Z(S)\backslash\{0\}$ and edges $\{(a, b) : ab = 0;\ a, b \in Z(S)^*, a \neq b\}$. Beck (1988) first presented this graph in the work on coloring of commutative rings. The graph was further studied by Anderson and Naseer (1993). These earlier works included zero in the vertex set of $\Gamma(S)$. Anderson & Livingston (1999) gave the definition of $\Gamma(S)$ which did not include zero in the vertex set. This work produced fundamental results on the graph $\Gamma(S)$. They determined when the graph $\Gamma(S)$ is complete or star and studied the automorphism group Aut$(\Gamma(S))$. The graph $\Gamma(S)$ has been extensively studied, and various authors have given new results and generalizations and also extended the graph to other algebraic structures. Generalizations of $\Gamma(S)$ are given in Anderson and McClurkin (2020), Safaeeyan *et al.* (2014) and DeMeyer *et al.* (2002). Structural properties of $\Gamma(S)$ for finite commutative rings are discussed by Singh and Bhat (2020). The connectivity, $\kappa(\Gamma(S))$, for finite rings S is discussed by Ju and Wu (2014) and Reddy *et al.* (2020). The clique number, $\omega(\Gamma(S))$, is given by Anderson *et al.* (2019) and Pirzada *et al.* (2020). The motivation for this research lies in the deeper exploration of the interplay between algebraic structures and graph-theoretic representations, which provides new insights into the structural behavior of rings and their ideals.

Redmond (2003) presented a generalization of $\Gamma(S)$ which is based on an ideal J of S. The vertices are $\{a \in S\backslash J\colon\ ab \in J$ for some $b \in S\backslash J\}$, where (a, b) is an edge if $ab \in J$. The graph is denoted by $\Gamma_J(S)$. He investigated how $\Gamma(S/J)$ and $\Gamma_J(S)$ are related and found the diameter, connectivity, clique number and girth of $\Gamma_J(S)$. Anderson and Naseer (1993) presented a list of rings S for which $\Gamma(S)$ has $1, 2, 3$ or 4 vertices. Redmond (2003) determined rings S such that $|V(\Gamma(S))| = 5$ and investigated when $\Gamma_J(S)$ is planar. Later on, the graph $\Gamma_J(S)$ has been studied by various authors (Maimani *et al.* 2006; Atani *et al.* 2011; Mehdi-Nezhad & Rahimi 2015; Mallika *et al.* 2017; Ansari-Toroghy *et al.* 2018). An ideal based quasi zero divisor graph has been defined by Yetkin Çelikel *et al.* (2021).

Nikmehr *et al.* (2021) defined the weakly zero divisor graph $W\Gamma(S)$ with $V(W\Gamma(S)) = Z(S)^*$, and $(a, b) \in E(W\Gamma(S))$ if $\exists\, r \in ann(a), s \in ann(b)$ such that $rs = 0$, where $ann(a) = \{m \in S :\ am = 0\}$. This graph contains $\Gamma(S)$ as a subgraph. The authors studied the diameter and girth of the graph $W\Gamma(S)$ and determined the rings for which $W\Gamma(S)$ is a star. They showed that the diameter of $W\Gamma(S)$ is at most 2 and the girth is $3, 4$ or ∞. They also determined when $\Gamma(S) = W\Gamma(S)$ and studied the coloring of $W\Gamma(S)$. Later on, the graph $W\Gamma(S)$ has been studied by various authors, see Nazim *et al.* (2023) and Rehman *et al.* (2023, 2024).

Ghafoor *et al.* (2024) introduced an ideal based weakly zero divisor graph of a commutative ring S, denoted by $W\Gamma_J(R)$, which contains $\Gamma_J(S)$ as a subgraph and is identical to $W\Gamma(S)$ when $J = \{0\}$. The authors explored the relationship between $W\Gamma(S/J)$ and $W\Gamma_J(S)$ and presented a method for constructing the graph $W\Gamma_J(S)$. Moreover, the authors determined when the graph $W\Gamma_J(S)$ is planar and classified rings S whose associated graph $W\Gamma_J(S)$ is planar.

In this article, the clique number and connectivity of $W\Gamma_J(S)$ are determined. Section 3.1 discusses the clique number of $W\Gamma_J(S)$. The connectivity and bridges are studied in Section 3.2. The ring S is commutative with $0 \neq 1$ and J is an ideal of S. Define the radical of an ideal J of the ring S by $\sqrt{J} = \{m \in S\colon\ m^k \in J$ for some $k \in \mathbb{N}\}$. For an element $u \in S$, $[u] = u + J = \{u + m :\ m \in J\}$ is an element of the factor ring S/J. A complete graph G is one in which all vertices are adjacent and it is denoted by K_n, where $n = |V(G)|$. A clique is a complete subgraph of a graph G. The clique number of G, $\omega(G)$, is the order of the largest clique in G, and $\omega(G) = \infty$ if $K_n \subseteq G\, \forall\, n \in \mathbb{N}$. A connected graph is the graph in which any two vertices are connected by a path. For a connected graph G, an $a \in V(G)$ is a cut point of G if $G - \{a\}$ is disconnected. If removal of at least n $(n \in \mathbb{N})$ vertices from G gives a disconnected graph, then n is the connectivity of G, denoted by $\kappa(G)$. A bridge is an edge whose removal disconnects the graph. For a graph G, let $d(u, v)$ be the length of the shortest path from u to v. The diameter of G, denoted by $diam(G)$, is $max\{d(u, v) :\ u, v \in V(G)\}$. The length of the shortest cycle in G is the girth of G, denoted by $gr(G)$ $(gr(G) = \infty$ if G is acyclic). For undefined graph theoretic terms, see Chartrand and Zhang (2013) and Bollobás (2012).

The idea of a clique finds application in network analysis, where cliques are used to compress networks (Buehrer & Chellapilla 2008) and to study temporal strong components of a dynamic network by computing the maximum clique for an associated graph (Rossi *et al.* 2015). In cheminformatics, clique-based approaches have been widely used for detecting large shared substructures among molecules (Gardiner *et al.* 1997; Raymond & Willett 2002). Another area of application with which graph theory is mainly concerned is connectivity in networks. Bunn *et al.* (2000) represented a landscape by a mathematical graph and used graph theoretic operations of vertex and edge removal to find whether the landscape is connected for two species sharing the same habitat, initiating the use of graph theory in landscape connectivity. From a scientific perspective, such investigations enrich the theoretical foundation that supports applications of algebraic graph theory in areas such as coding theory, cryptography, and network analysis, where the structural properties of graphs derived from algebraic systems play a crucial role.

2. Preliminary Result

This section contains the definitions and results related to this research. The definition of an ideal based weakly zero divisor graph is as follows.

Definition 2.1. (Ghafoor *et al.* 2024) The ideal based weakly zero divisor graph of S, denoted by $W\Gamma_J(S)$, is an undirected graph with vertices $\{a \in S\backslash J : ab \in J \text{ for some } b \in S\backslash J\}$, where distinct vertices a and b are adjacent if and only if there exist $r \in (J : a)$ and $s \in (J : b)$ such that $rs \in J$, where $(J : a) = \{r \in S : ra \in J\}$.

The example of $W\Gamma_J(S)$ is shown below.

Example 2.2. Let $S = \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $J = \mathbb{Z}_2 \times \{0\}$. The vertices of $W\Gamma_J(S)$, listed in order from 1 to 14, are $(0,2), (1,2), (0,4), (1,4), (0,6), (1,6), (0,8), (1,8), (0,10), (1,10), (0,3), (1,3), (0,9), (1,9)$. Figure 1 depicts the graph of $W\Gamma_J(S)$. We can see that $W\Gamma_J(S) \neq \Gamma_J(S)$, for example, $((0,2),(0,3)) \in E(W\Gamma_J(S))$ but $((0,2),(0,3)) \notin E(\Gamma_J(S))$.

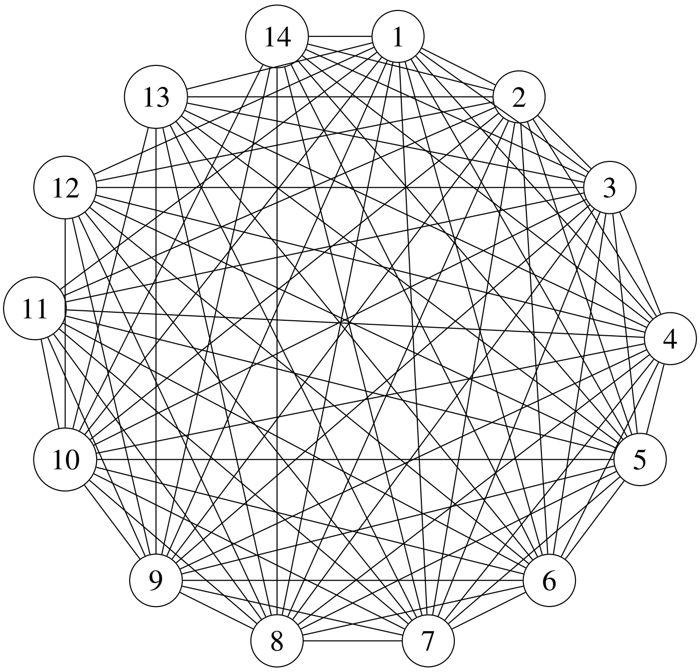

Figure 1: $W\Gamma_J(S), S = \mathbb{Z}_2 \times \mathbb{Z}_{12}, J = \mathbb{Z}_2 \times \{0\}$

The following result shows when $W\Gamma_J(S)$ is empty and when it is identical to $W\Gamma(R)$.

Proposition 2.3. (Ghafoor *et al.* 2024, Proposition 19) *Let J be an ideal of S.*

(1) If $J = \{0\}$, then $W\Gamma_J(S) = W\Gamma(S)$.

(2) If $J \neq \{0\}$, then $W\Gamma_J(S) = \emptyset$ iff J is a prime ideal of S.

Thus, $W\Gamma_J(S) = \emptyset$ iff S/J is an integral domain iff $W\Gamma(S/J) = \emptyset$. The following theorem shows how $W\Gamma_J(S)$ and $W\Gamma(S/J)$ are related.

Theorem 2.4. (Ghafoor *et al.* 2024, Theorem 22) *Let $u, v \in S\backslash J$. Then:*

(1) If $([u],[v]) \in E(W\Gamma(S/J))$, then $(u,v) \in E(W\Gamma_J(S))$.

(2) If $(u,v) \in E(W\Gamma_J(S))$ and $[u] \neq [v]$, then $([u],[v]) \in E(W\Gamma(S/J))$.

(3) If $(u,v) \in E(W\Gamma_J(S))$ and $[u] = [v]$, then $\exists\, r,s \in (J:u)$, where $(J:u) = (J:v)$, such that $rs \in J$.

A direct consequence of Theorem 2.4 gives the following corollary.

Corollary 2.5. (Ghafoor *et al.* 2024, Corollary 23) *If $(u,v) \in E(W\Gamma_J(S))$, $[u] \neq [v]$ then $(u+l, v+m) \in E(W\Gamma_J(S))\,\forall\, l,m \in J$. If $\exists\, r,s \in (J:u)$ such that $rs \in J$, then $(u+l, u+m) \in E(W\Gamma_J(S))\,\forall\, l,m \in J, l \neq m$.*

The next result states that $W\Gamma(S/J)$ is isomorphic to a subgraph of $W\Gamma_J(S)$.

Proposition 2.6. (Ghafoor *et al.* 2024, Proposition 24) *Let J be an ideal of S. The graph $W\Gamma_J(S)$ has $|J|$ disjoint subgraphs isomorphic to $W\Gamma(S/J)$.*

As in Ghafoor *et al.* (2024), the graph $W\Gamma_J(S)$ can be constructed by the method that follows. Let $\{u_\alpha : u_\alpha + J \in V(W\Gamma(S/J)), \alpha \in \Lambda\} \subseteq S$ and let $G_l (l \in J)$ be the subgraph of $W\Gamma_J(S)$ with vertices $\{u_\alpha + l : \alpha \in \Lambda\}$, where $(u_\alpha + l, u_\gamma + l) \in E(G_l)$ if and only if $(u_\alpha + J, u_\gamma + J) \in E(W\Gamma(S/J)$. Define the graph G with vertices $V(G) = \bigcup_{l \in J} V(G_l)$. $E(G)$ is defined to be:

1. $(u_\alpha + l, u_\gamma + l) \in E(G_l)$ for each $l \in J$,
2. for $\alpha, \gamma \in \Lambda, \alpha \neq \gamma$ and $l,m \in J$, $(u_\alpha + l, u_\gamma + m) \in E(G)$ iff $([u_\alpha],[u_\gamma]) \in E(W\Gamma(S/J))$ (that is, $\exists\, r \in (J:u_\alpha), s \in (J:u_\gamma)$ such that $rs \in J$),
3. for $\alpha \in \Lambda$ and $l,m \in J, l \neq m$, $(u_\alpha + l, u_\alpha + m) \in E(G)$ iff $\exists\, r,s \in (J:u_\alpha)$ such that $rs \in J$.

In this article, the method for constructing $W\Gamma_J(S)$ is used together with Theorem 2.4 and Corollary 2.5 to find $\omega(W\Gamma_J(S))$ and its relationship with $\omega(W\Gamma(S/J))$. The following result is used to find $|V(W\Gamma_J(S))|$.

Remark 2.7. (Ghafoor *et al.* 2024, Remark 25) Let J be an ideal of S. Then $V(\Gamma_J(S)) = V(W\Gamma_J(S))$ and $|V(\Gamma_J(S))| = |V(W\Gamma_J(S))| < \infty$ if and only if $|S| < \infty$ or J is prime. Moreover, if $|V(W\Gamma(S/J))| = n$, then $|V(W\Gamma_J(S))| = n \cdot J$.

The next lemma gives some properties of $W\Gamma_J(S)$ that are used to find the clique number and connectivity of $\Gamma_J(S)$.

Lemma 2.8. (Ghafoor *et al.* 2024, Lemma 20) *Let J be an ideal of S.*

(1) If $(u,v) \in E(\Gamma_J(S))$, then $(u,v) \in E(W\Gamma_J(S))$.

(2) If $u \in \sqrt{J}\backslash J$, then $(u,v) \in E(W\Gamma_J(S))\,\forall\, v \in V(W\Gamma_J(S))$.

(3) $\sqrt{J}\backslash J$ is a complete subgraph of $W\Gamma_J(S)$.

The following theorem gives the diameter and girth of the graph $W\Gamma_J(S)$.

Theorem 2.9. (Ghafoor *et al.* 2024, Theorem 21) *Let J be an ideal of S. Then $W\Gamma_J(S)$ is connected with $diam(W\Gamma_J(S)) \leq 2$. If $gr(W\Gamma_J(S)) \neq \infty$, then $gr(W\Gamma_J(S)) \leq 4$.*

The following theorem from Anderson and Livingston (1999) is used to find $\omega(W\Gamma_J(S))$ when $|V(W\Gamma(S/J))| \geq 2$.

Theorem 2.10. (Anderson & Livingston 1999, Theorem 2.3) *Let S be a commutative ring. Then $\Gamma(S)$ is connected and $diam(\Gamma(S)) \leq 3$. Moreover, if $\Gamma(S)$ contains a cycle, then $gr(\Gamma(S)) \leq 7$.*

The following lemma gives some properties of $\Gamma_J(S)$ when $\Gamma(S/J)$ is complete or a star graph and is used to find $\omega(W\Gamma_J(S))$ when $|V(W\Gamma(S/J))| = 3$.

Lemma 2.11. (Mehdi-Nezhad & Rahimi 2015, Lemma 4.10) *Let J be a proper ideal of a ring S. Then:*

(1) If $\Gamma(S/J)$ is a complete graph on $n = 1$ or $n \geq 3$ vertices, then each vertex of $\Gamma(S/J)$ is a connected column in $\Gamma_J(S)$. Moreover, $\Gamma_J(S)$ is a complete graph on $n \cdot |J|$ vertices.

(2) If $\Gamma(S/J)$ has only two vertices, then either both or none of the vertices are connected columns in $\Gamma_J(S)$.

(3) If $\Gamma(S/J)$ is a star graph with three vertices, then it has no connected columns or only the center can be a connected column.

(4) If $\Gamma(S/J) \cong K_{1,3}$, then it has no connected columns.

(5) If $\Gamma(S/J)$ is a star graph of size $n \geq 3$, then none of its non-central vertices is a connected column of $\Gamma_J(S)$.

Next, the following result from Redmond (2003) is used together with Lemma 2.8 to find whether $W\Gamma_J(S)$ has a cut-point.

Theorem 2.12. (Redmond 2003, Theorem 3.2) *If J is a nonzero proper ideal of S, then $\Gamma_J(S)$ has no cut-points.*

The following theorem from Redmond (2003) is used together with Corollary 2.5 and Lemma 2.8 to find the bounds on $\kappa(W\Gamma_J(S))$.

Theorem 2.13. (Redmond 2003, Theorem 3.3) *Let J be a nonzero proper ideal of a ring S.*

(1) If $|V(\Gamma(S/J))| = 1$, then $\kappa(\Gamma_J(S)) = |J| - 1$.

(2) If $2 \leq |V(\Gamma(S/J))|$, then $2 \leq \kappa(\Gamma_J(S)) \leq (|J| \cdot \kappa(\Gamma(S/J)))$.

(3) $\kappa(\Gamma_J(S)) \geq |J| - 1$.

Finally, by using the following result from Redmond (2003) together with Lemma 2.8, the necessary and sufficient conditions are found under which $W\Gamma_J(S)$ has a bridge.

Proposition 2.14. (Redmond 2003, Proposition 3.5) *Let J be an ideal of a ring S. Then $\Gamma_J(S)$ has a bridge iff either*

(1) $|V(\Gamma_J(S))| = 2$, or

(2) $J = \{0\}$ and $\Gamma(S)$ has a bridge.

3. Results and Discussion

In this section, the results on $\omega(W\Gamma_J(S))$ and $\kappa(W\Gamma_J(S))$ are presented. The bounds on $\omega(W\Gamma_J(S))$ and $\kappa(W\Gamma_J(S))$ are determined.

3.1. *Clique number of* $W\Gamma_J(S)$

In this subsection, $\omega(W\Gamma_J(S))$ is determined and its relationship with $\omega(W\Gamma(S/J))$ is investigated. The bound on $\omega(W\Gamma_J(S))$ is also found. Note that $\omega(W\Gamma(S/J)) \leq \omega(W\Gamma_J(S)))$ and that the equality holds when $\omega(W\Gamma(S/J)) = \infty$.

The equality might hold in the cases discussed in the following few results.

Proposition 3.1. *If $u^2 \in J$ for $u \in S \backslash J$, then $\omega(W\Gamma_J(S)) \geq |J|$.*

Proof. If $u^2 \in J$ for $u \in S \backslash J$, then $(u+l)(u+m) \in J \, \forall \, l, m \in J, l \neq m$. Thus, $(u+l, u+m) \in E(\Gamma_J(S))$. By Lemma 2.8, $(u+l, u+m) \in E(W\Gamma_J(S))$. □

Corollary 3.2. *If $u^2 \in J$ for $u \in S \backslash J$ and $|J| = \infty$, then $\omega(W\Gamma_J(S)) = \infty$.*

Corollary 3.3. *If $|V(W\Gamma(S/J))| = 1$, then $\omega(W\Gamma_J(S)) = |J|$. Thus, if $J \neq \{0\}$, then $\omega(W\Gamma(S/J)) < \omega(W\Gamma_J(S))$.*

Proposition 3.4. *If $u^2 \in J$ for $u \in S \backslash J$ and $|V(W\Gamma(S/J))| \geq 2$, then $\omega(W\Gamma_J(S)) \geq 2|J|$.*

Proof. Let $u^2 \in J$ and $2 \leq |V(W\Gamma(S/J))|$. By Theorem 2.10, $\Gamma(S/J)$ is connected. Since $|V(\Gamma(S/J))| \geq 2$, $\exists \, v \in S \backslash J$ such that $[u] \neq [v]$ and $([u], [v]) \in E(\Gamma(S/J))$ (i.e., $uv \in J$). By Lemma 2.8, $(u, v) \in E(W\Gamma_J(S))$. Also, since $u \in (J : v)$ and $u^2 \in J$, we have $(v+l, v+m) \in E(W\Gamma_J(S)) \, \forall \, l, m \in J, l \neq m$, by Corollary 2.5. Thus, $[u] \cup [v]$ forms a clique of $W\Gamma_J(S)$. □

Proposition 3.5. *If $u^2 \in J$ for $u \in S \backslash J$ and $|V(W\Gamma(S/J))| = 3$, then $\omega(W\Gamma_J(S)) = 3|J|$. Thus, if $\{0\} \neq J$, then $\omega(W\Gamma_J(S)) > \omega(\Gamma(S/J))$.*

Proof. Let $u^2 \in J$ and $|V(W\Gamma(S/J)| = 3$. Since $V(W\Gamma(S/J)) = V(\Gamma(S/J))$, the graph $\Gamma(S/J)$ has three vertices. Suppose that $\Gamma(S/J) \cong K_3$. By Lemma 2.11, $\Gamma_J(S) \cong K_{3|J|}$. Thus, by Lemma 2.8, $\omega(W\Gamma_J(S)) = 3|J|$. Assume that $\Gamma(S/J) \ncong K_3$. By Theorem 2.10, $\Gamma(S/J)$ is connected. Since $\Gamma(S/J) \ncong K_3$ and $\Gamma(S/J)$ is connected, it follows that $\Gamma(S/J) \cong K_{1,2}$. Thus, $[u]$ is the center of $\Gamma(S/J)$ by Lemma 2.11. Let $[v]$ and $[w]$ be the other two vertices of $\Gamma(S/J)$. Then $(([u], [v]), ([u], [w]) \in E(\Gamma(S/J))$ and so $uv, uw \in J$. By Lemma 2.8, $(u, v), (u, w) \in E(W\Gamma_J(S))$. Also, $(v, w) \in E(W\Gamma_J(S))$ since $u \in (J : v) \cap (J : w)$ and $u^2 \in J$. By Corollary 2.5, $W\Gamma_J(S) \cong K_{3|J|}$. □

Proposition 3.6. *If $gr(\Gamma(S/J)) = 3$, then $\omega(W\Gamma_J(S)) \geq 3|J|$.*

Proof. Let $[u] - [v] - [w] - [u]$ be a triangle in $\Gamma(S/J)$. Then $uv \in J$, $vw \in J$ and $uw \in J$. By Lemma 2.8, each of u, v and w is adjacent to the other two in $W\Gamma_J(S)$. By Corollary 2.5, we have that $(u+l, v+m) \in E(W\Gamma_J(S))$, $(u+l, w+m) \in E(W\Gamma_J(S))$ and $(v+l, w+m) \in E(W\Gamma_J(S)) \, \forall \, l, m \in J$. Since $v, w \in (J : u)$ and $vw \in J$, it follows by Corollary 2.5 that $(u+l, u+m) \in E(W\Gamma_J(S)) \, \forall \, l, m \in J$, $l \neq m$. Similarly, $(v+l, v+m) \in E(W\Gamma_J(S))$ and $(w+l, w+m) \in E(W\Gamma_J(S)) \, \forall \, l, m \in J$, $l \neq m$. Thus, $[u] \cup [v] \cup [w]$ forms a clique of $W\Gamma_J(S)$. □

Corollary 3.7. *If $gr(\Gamma(S/J)) = 3$ and J is infinite, then $\omega(W\Gamma_J(S)) = \infty$.*

Proposition 3.8. *If $|V(\Gamma(S/J))| \geq 4$ and $gr(\Gamma(S/J)) = 3$, then $\omega(W\Gamma_J(S)) \geq 3|J| + 1$.*

Proof. Let $[u] - [v] - [w] - [u]$ be a triangle in $\Gamma(S/J)$. By the proof of Proposition 3.6, $[u] \cup [v] \cup [w]$ forms a clique of $W\Gamma_J(S)$. Since $|V(\Gamma(S/J))| \geq 4$ and $\Gamma(S/J)$ is connected, $\exists \, s \in S \backslash J$ such that $[s] \neq [u], [s] \neq [v], [s] \neq [w]$ and $su \in J, sv \in J$ or $sw \in J$. Suppose that $su \in J$. By Lemma 2.8, $(s, u) \in E(W\Gamma_J(S))$. Since $su \in J, vw \in J$ and $uw \in J$, we have $(s, v) \in E(W\Gamma_J(S))$. Also, $(s, w) \in E(W\Gamma_J(S))$ since $su \in J, vw \in J$ and $uv \in J$. By Corollary 2.5, the subgraph induced by $[u] \cup [v] \cup [w] \cup \{s\}$ forms a clique of $W\Gamma_J(S)$. We get the same result when $sv \in J$ or $sw \in J$. □

Theorem 3.9. *Let J be an ideal of S. Then $\omega(W\Gamma_J(S)) \leq |J| \cdot \omega(W\Gamma(S/J))$.*

Proof. Let $\omega(W\Gamma(S/J)) = k$. Let the set $H = \{[a_1], [a_2], \ldots, [a_k]\} \subseteq V(W\Gamma(S/J))$ induce a maximal clique of $W\Gamma(S/J)$. Let $T = \{a+m : [a] \in H, m \in J\}$. Suppose that for each $[a] \in H, \exists \, r, s \in (J : a)$ such that $rs \in J$. Then, by Corollary 2.5, the set T induces a clique of $W\Gamma_J(S)$. If $T \cup \{p\}$ is a clique of $W\Gamma_J(S)$, then $(p, a+m) \in E(W\Gamma_J(S))$. Thus, $\exists \, u \in (J : p), v \in (J : a+m)$ such that $uv \in J$. Then $[p][u] = [a][v] = [u][v] = [0] = J$ and so $([p], [a]) \in E(W\Gamma(S/J))$. So $H \cup \{[p]\}$ is a clique of $W\Gamma(S/J)$, a contradiction. Thus, T induces a maximal clique and so $\omega(W\Gamma_J(S)) = |T|$. In all other cases, $\omega(W\Gamma_J(S)) < |J| \cdot \omega(W\Gamma(S/J))$. □

Example 3.10. Let $S = \mathbb{Z}_4 \times \mathbb{Z}_6$ and $J = \{(0,0),(0,3)\}$. The vertices of $W\Gamma(S/J)$, listed in order from 1 to 7, are $[(1,0)], [(0,1)], [(0,2)], [(2,0)], [(2,1)], [(2,2)], [(3,0)]$. The set $\{[(0,1)], [(0,2)], [(2,0)], [(2,1)], [(2,2)], [(3,0)]\}$ is the largest clique of $W\Gamma(S/J)$. The set $\{[(0,1)], [(0,2)], [(1,0)], [(2,0)], [(2,1)], [(2,2)]\}$ is also the largest clique of $W\Gamma(S/J)$. Thus, $\omega(W\Gamma(S/J)) = 6$. The graph $W\Gamma(S/J)$ is given in Figure 2. The vertices of $W\Gamma_J(S)$, listed in order from 1 to 14, are $(0,1), (0,2), (0,4), (0,5), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (1,0), (1,3), (3,0), (3,3)$. The following two sets form the largest cliques of $W\Gamma_J(S)$:

$$\{(0,1),(0,2),(0,4),(0,5),(1,0),(2,0),(2,1),(2,2),(2,3),(2,4),(2,5)\},$$

$$\{(0,1),(0,2),(0,4),(0,5),(2,0),(2,1),(2,2),(2,3),(2,4),(2,5),(3,0)\}.$$

Thus, $\omega(W\Gamma_J(S)) = 11$. The graph $W\Gamma_J(S)$ is isomorphic to the graph in Figure 1. Note that $\omega(W\Gamma_J(S)) = 11 < |J| \cdot \omega(W\Gamma(S/J)) = 12$.

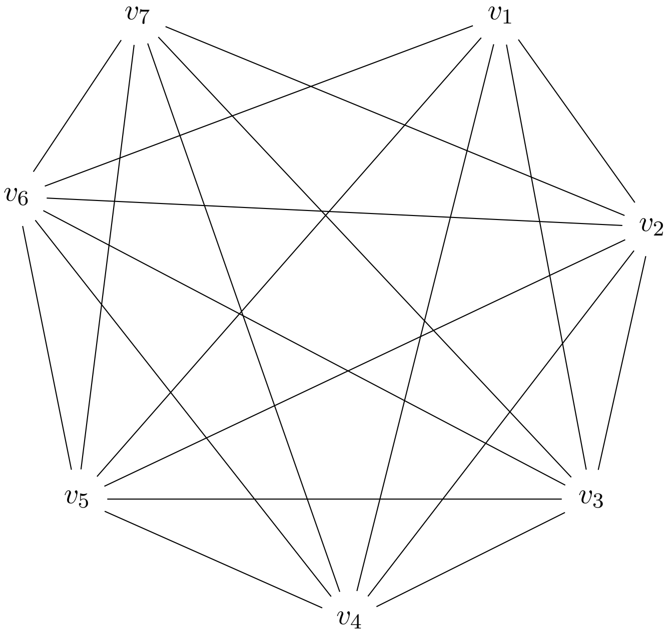

Figure 2: $W\Gamma(S/J), S = \mathbb{Z}_4 \times \mathbb{Z}_6, J = \{(0,0),(0,3)\}$

Theorem 3.11. *If* $gr(W\Gamma_J(S)) = 4$, *then* $\omega(W\Gamma_J(S)) = \omega(W\Gamma(S/J))$.

Proof. Assume that $gr(W\Gamma_J(S)) = 4$. Then $\sqrt{J} = J$ by part (2) of Lemma 2.8 (otherwise $gr(W\Gamma_J(S)) = 3$). Let $\omega(W\Gamma(S/J)) = N < \infty$. Suppose $G = \{a_1, a_2, ..., a_{N+1}\}$ is a clique in $W\Gamma_J(S)$. Then $W\Gamma(S/J)$ has a complete subgraph on the vertices $[a_1], [a_2], ...[a_{N+1}]$. But $\omega(W\Gamma(S/J)) = N$ implies $[a_j] = [a_k]$, for some $j \neq k$. Since $(a_j, a_k) \in E(W\Gamma_J(S))$, $\exists\, r \in (J : a_j), s \in (J : a_k)$ such that $rs \in J$. But then $[a_j s] = [a_j][s] = [a_k][s] = [a_k s] = J$, which implies that $a_j s \in J$. Since $\sqrt{J} = J$, it follows that $a_j \neq r, a_j \neq s$ and $r \neq s$. Thus, $a_j - r - s - a_j$ is a triangle in $W\Gamma_J(S)$, a contradiction. □

Example 3.12. Let $S = \mathbb{Z}_8 \times \mathbb{Z}_4$, $J = \{(0,0),(0,2),(2,0),(2,2),(4,0),(4,2),(6,0),(6,2)\}$. The vertices of $W\Gamma_J(S)$ are $(0,1), (0,3), (1,0), (1,2), (2,1), (2,3), (3,0), (3,2), (4,1), (4,3), (5,0), (5,2), (6,1), (6,3), (7,0), (7,2)$. $W\Gamma_J(S) = K_{8,8}$ with partite sets

$$A = \{(0,1),(0,3),(2,1),(2,3),(4,1),(4,3),(6,1),(6,3)\}$$

and

$$B = \{(1,0),(1,2),(3,0),(3,2),(5,0),(5,2),(7,0),(7,2)\}.$$

Therefore, $\omega(W\Gamma_J(S)) = 2$ and $gr(W\Gamma_J(S)) = 4$. Let v_1 to v_8 denote the vertices of $W\Gamma_J(S)$ in the set A and v_9 to v_{16} denote the vertices in the set B. The graph $W\Gamma_J(S)$ is given in Figure 3. The graph $W\Gamma(S/J)$ has vertices $[(0,1)]$ and $[(1,0)]$. Since $|V(W\Gamma(S/J))| = 2$, we have $\omega(W\Gamma(S/J)) = 2$.

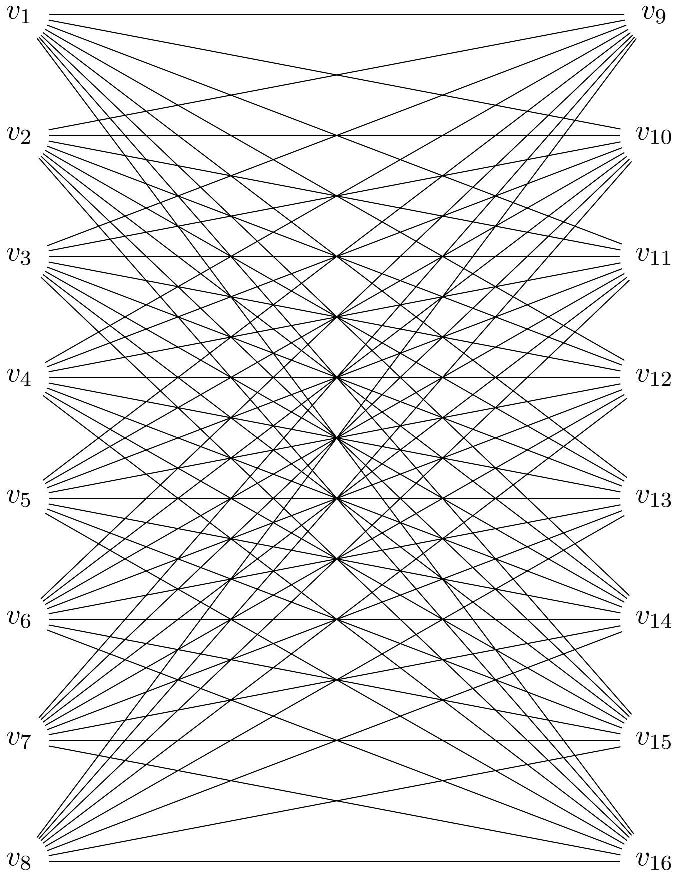

Figure 3: $W\Gamma_J(S), S = \mathbb{Z}_8 \times \mathbb{Z}_4, J = \{(0,0),(0,2),(2,0),(2,2),(4,0),(4,2),(6,0),(6,2)\}$

Theorem 3.13. *Let* $a \in \sqrt{J}\backslash J$. *Then* $(n-1) \leq \omega(W\Gamma_J(S))$.

Proof. Since $a \in \sqrt{J}$, there exists $n \in \mathbb{N}$ such that $a^n \in J$ and $a^q \notin J\,\forall\, q,\ 1 \leq q \leq n-1$. Then $[a^p] \neq [a^q]$ for $1 \leq p, q \leq n-1$ (otherwise $a^p \in J$ for some p, $1 \leq p \leq n-1$). Also $a^q \in \sqrt{J}\,\forall\, q,\ 1 \leq q \leq n-1$. By Lemma 2.8,

$$(a^p, a^q) \in E(W\Gamma_J(S))\ \forall\, p, q\ (1 \leq p, q \leq n-1,\ p \neq q).$$

Clearly for each $p \in \{1, 2, \ldots, n-1\}, \exists\, q, r \in \{1, 2, \ldots, n-1\}$ such that $a^q, a^r \in (J : a^p)$ and $a^q a^r \in J$. By Corollary 2.5, $[a] \cup [a^2] \cup \cdots \cup [a^{n-1}]$ forms a clique of $W\Gamma_J(S)$. $\square$

3.2. *Connectivity of* $W\Gamma_J(S)$

In this subsection, the connectivity of $W\Gamma_J(S)$ is found and its relationship with the connectivity of $W\Gamma(S/J)$ is investigated. The bridges in $W\Gamma_J(S)$ are also discussed.

Proposition 3.14. *If J is a nonzero proper ideal of S, then $W\Gamma_J(S)$ has no cut-points.*

Proof. By Theorem 2.12, $\Gamma_J(S)$ has no cut-point. Since $V(W\Gamma_J(S)) = V(\Gamma_J(S))$ and $\Gamma_J(S)$ has no cut-point, it follows from part (1) of Lemma 2.8 that $W\Gamma_J(S)$ has no cut point. □

We find bounds on $\kappa(W\Gamma_J(S))$.

Theorem 3.15. *Let J be a nonzero proper ideal of a ring S.*

(1) *If $|V(W\Gamma(S/J))| = 1$, then $\kappa(W\Gamma_J(S)) = |J| - 1$.*

(2) *If $2 \leq |V(W\Gamma(S/J))|$, then $2 \leq \kappa(W\Gamma_J(S)) \leq (|J| \cdot \kappa(W\Gamma(S/J)))$.*

(3) *$\kappa(W\Gamma_J(S)) \geq \kappa(\Gamma_J(S)) \geq |J| - 1$.*

Proof.

(1) If $|V(W\Gamma(S/J))| = 1$, then $W\Gamma_J(S) = K_{|J|}$.

(2) Since $W\Gamma_J(S)$ is connected, $1 \leq \kappa(W\Gamma_J(S))$. Let $k = \kappa(W\Gamma(S/J))$. Let removal of the vertices $[a_1], [a_2], ..., [a_k]$ of $W\Gamma(S/J)$ give a disconnected graph. Let removal of elements in $[a_i], 1 \leq i \leq k$, from $V(W\Gamma_J(S))$ give the graph G (this means that $k \cdot |J|$ vertices have been removed). Since removal of $[a_1], [a_2], ..., [a_k]$ from $W\Gamma(S/J)$ gives a disconnected graph, $\exists\, [b], [c] \in V(W\Gamma(S/J))$ such that $[b]$ and $[c]$ are disconnected after this removal. Then $b, c \in V(G)$. Suppose $b - u_1 - \cdots - u_m - c$ is a path in G. Thus, $[u_s] \neq [u_{s+1}]$ for $1 \leq s \leq m$ considering Corollary 2.5. Then $[b] - [u_1] - \cdots - [u_m] - [c]$ is a path in $W\Gamma(S/J)$ after the removal of $[a_1], [a_2], ..., [a_k]$, a contradiction. Thus, $\kappa(W\Gamma_J(S)) \leq k \cdot |J|$.

(3) Let $k = \kappa(W\Gamma_J(S))$ and let removal of the vertices $a_1, a_2, ..., a_k$ of $W\Gamma_J(S)$ give a disconnected graph. Since $V(W\Gamma_J(S)) = V(\Gamma_J(S))$ and $\Gamma_J(S) \subseteq W\Gamma_J(S)$, the removal of vertices $a_1, a_2, ..., a_k$ from $\Gamma_J(S)$ must give a disconnected graph. Hence $\kappa(\Gamma_J(S)) \leq k$. Also, $|J| - 1 \leq \kappa(\Gamma_J(S))$ by Theorem 2.13. □

Corollary 3.16. *If J is a nonzero proper ideal of S and J is non-prime, then $|J| - 1 \leq \kappa(W\Gamma_J(S)) \leq |J| \cdot \kappa(W\Gamma(S/J))$. In particular, $\kappa(W\Gamma_J(S)) = \infty$ if $|J| = \infty$.*

Example 3.17.

(1) Let $S = \mathbb{Z}_4 \times \mathbb{Z}_6$, $J = \{(0,0), (0,3)\}$. The vertices of $W\Gamma(S/J)$, listed in order from 1 to 7, are $[(1,0)], [(0,1)], [(0,2)], [(2,0)], [(2,1)], [(2,2)], [(3,0)]$. The graph $W\Gamma(S/J)$ is given in Figure 2. Consider the set

$$A = \{[(0,1)], [(0,2)], [(2,0)], [(2,1)], [(2,2)]\},$$

then $W\Gamma(S/J) \backslash A$ is disconnected and $W\Gamma(S/J) \backslash B$ is not disconnected for any $B \subseteq V(W\Gamma(S/J))$ with $|B| < |A|$. Thus, $\kappa(W\Gamma(S/J)) = 5$. The vertices of $W\Gamma_J(S)$, listed in order from 1 to 14, are $(0,1), (0,2), (0,4), (0,5), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (1,0), (1,3), (3,0), (3,3)$. The graph $W\Gamma_J(S)$ is shown in Figure 1. Consider the set

$$B = \{(0,1), (0,2), (0,4), (0,5), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5)\},$$

then $W\Gamma_J(S) \backslash B$ is disconnected and $W\Gamma(S/J) \backslash C$ is not disconnected for any $C \subseteq V(W\Gamma_J(S))$ with $|C| < |B|$. Thus, $\kappa(W\Gamma_J(S)) = 10$. Note that

$$|J| \cdot \kappa(W\Gamma(S/J)) = (2)(5) = 10 = \kappa(W\Gamma_J(S)).$$

(2) Let $S = \mathbb{Z}_8 \times \mathbb{Z}_4$, $J = \{(0,0),(0,2),(2,0),(2,2),(4,0),(4,2),(6,0),(6,2)\}$. Then $W\Gamma_J(S) = K_{8,8}$ with partite sets

$$A = \{(0,1),(0,3),(2,1),(2,3),(4,1),(4,3),(6,1),(6,3)\}$$

and

$$B = \{(1,0),(1,2),(3,0),(3,2),(5,0),(5,2),(7,0),(7,2)\}.$$

Since $W\Gamma_J(S)\backslash A$ and $W\Gamma_J(S)\backslash B$ are disconnected and $|A| = |B| = 8$, we have $\kappa(W\Gamma_J(S)) = 8 = |J|$.

(3) Let $S = \mathbb{Z}_{24}$ and $J = \langle 8 \rangle$, that is, the ideal J of S generated by 8. Then the vertex set of $W\Gamma_J(S)$ is

$$V(W\Gamma_J(S)) = \{2,4,6,10,12,14,18,20,22\}.$$

The graph $W\Gamma_J(S)$ is complete and so $\kappa(W\Gamma_J(S)) = 8$. The graph $\Gamma_J(S)$ is not complete, as shown in Figure 4. Let $A = \{4,12,20\}$. Then $\Gamma_J(S)\backslash A$ is disconnected and so $\kappa(\Gamma_J(S)) = |A| = 3$. Thus, $\kappa(\Gamma_J(S)) < \kappa(W\Gamma_J(S))$.

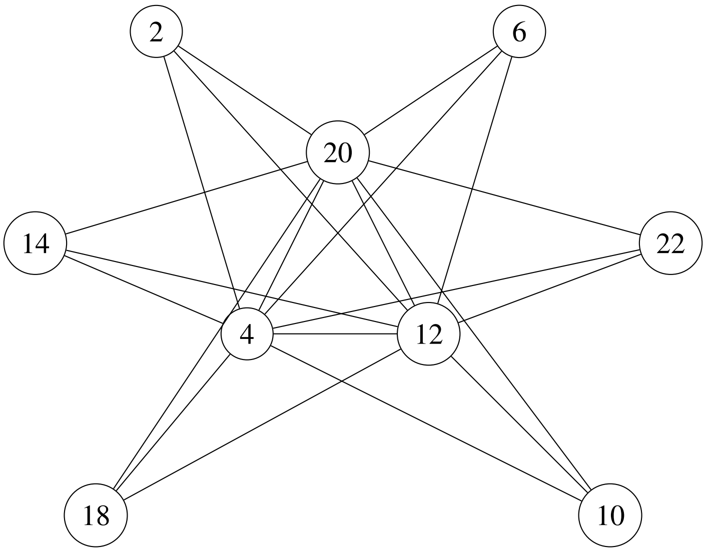

Figure 4: $\Gamma_J(S), S = \mathbb{Z}_{24}, J = <8>$

Proposition 3.18. *The graph $W\Gamma_J(S)$ has a bridge iff either*

(1) $|V(W\Gamma_J(S))| = 2$*, or*

(2) $J = \{0\}$ *and* $W\Gamma(S)$ *has a bridge.*

Proof. If (1) or (2) hold, then $W\Gamma_J(S)$ has a bridge. Conversely, assume that $W\Gamma_J(S)$ has a bridge. Since $\Gamma_J(S)$ is connected subgraph of $W\Gamma_J(S)$ and $V(\Gamma_J(S)) = V(W\Gamma_J(S))$, the removal of $e \in E(W\Gamma_J(S))\backslash E(\Gamma_J(S))$ from $W\Gamma_J(S)$ does not give a disconnected graph (the removal of an edge from a graph having a connected subgraph with the same vertices does not disconnect the graph without disconnecting the subgraph and the removal of e from $W\Gamma_J(S)$ does not disconnect the subgraph $\Gamma_J(S)$ of $W\Gamma_J(S)$ since it is not an edge of $\Gamma_J(S)$). Thus, if e is a bridge of $W\Gamma_J(S)$, then e must be a bridge of $\Gamma_J(S)$. By Proposition 2.14, $\Gamma_J(S)$ has a bridge iff either (a) $|V(\Gamma_J(S))| = 2$ or (b) $J = \{0\}$ and $\Gamma(S)$ has a bridge. If (a) holds, then $|V(W\Gamma_J(S))| = 2$ since $V(W\Gamma_J(S)) = V(\Gamma_J(S))$. If (b) holds, then $W\Gamma_J(S) = W\Gamma(S)$ by Proposition 2.3 since $J = \{0\}$. Thus, either (1) or (2) hold. □

4. Conclusion

Our results showed that $\omega(W\Gamma(S/J))$ is less than or equal to $\omega(W\Gamma_J(S))$ and the equality holds when $\omega(W\Gamma(S/J)) = \infty$ or $gr(W\Gamma_J(S)) = 4$. It was shown that $\omega(W\Gamma_J(S)) \leq |J| \cdot \omega(W\Gamma(S/J))$. Also, it was shown that if $2 \leq |V(W\Gamma(S/J))|$, then $|J| - 1 \leq \kappa(W\Gamma_J(S)) \leq (|J| \cdot \kappa(W\Gamma(S/J)))$. The results showed that if $W\Gamma_J(S)$ has a bridge, then either (i) $|V(W\Gamma_J(S))| = 2$ or (ii) $J = \{0\}$ and $W\Gamma(S)$ has a bridge. These results give new information on the graph $W\Gamma_J(S)$ and reveal interesting relationships that hold between the graphical properties of $W\Gamma_J(S)$ and $W\Gamma(S/J)$.

From the results that have been obtained, it is clear that the graph $W\Gamma_J(S)$ presents an opportunity to further explore the relationships between its properties and the properties of some of the existing graphs. Moreover, there is a possibility of studying $W\Gamma_J(S)$ with respect to various types of ideal J of S which will improve our understanding of the structure of the ring and the structure of the graph.

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