CHAPTER 2
DIFFERENTIATION

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The Geometrical Meaning of Differentiation

The Geometrical Meaning of Differentiation

Definition 2.1: The Derivative

Let $y = f(x)$ be a function. The derivative of a function f with respect to x, denoted by f', is defined by

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

provided the limit exists.

Note that the domain of f' is a subset of the domain of f in which the limit exists. The domain of f' may be smaller than the domain of f. If f' exists, then we say that f is differentiable at x.

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The Geometrical Meaning of Differentiation

Definition 2.2: Gradient of the Tangent

*If $P(x_0, y_0)$ is a point on a curve $y = f(x)$, then the **gradient of the tangent** to the curve at P is defined as*

$$m = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

provided the limit exists.

The Geometrical Meaning of Differentiation

The First Principle

Step 1 Given $y = f(x)$. Write the expression $f(x+\delta x)$.

Step 2 Obtain the expression difference between $f(x+\delta x)$ and $f(x)$, that is

$$f(x+\delta x) - f(x)$$

Step 3 Simplify the expression

$$\frac{f(x+\delta x) - f(x)}{\delta x}$$

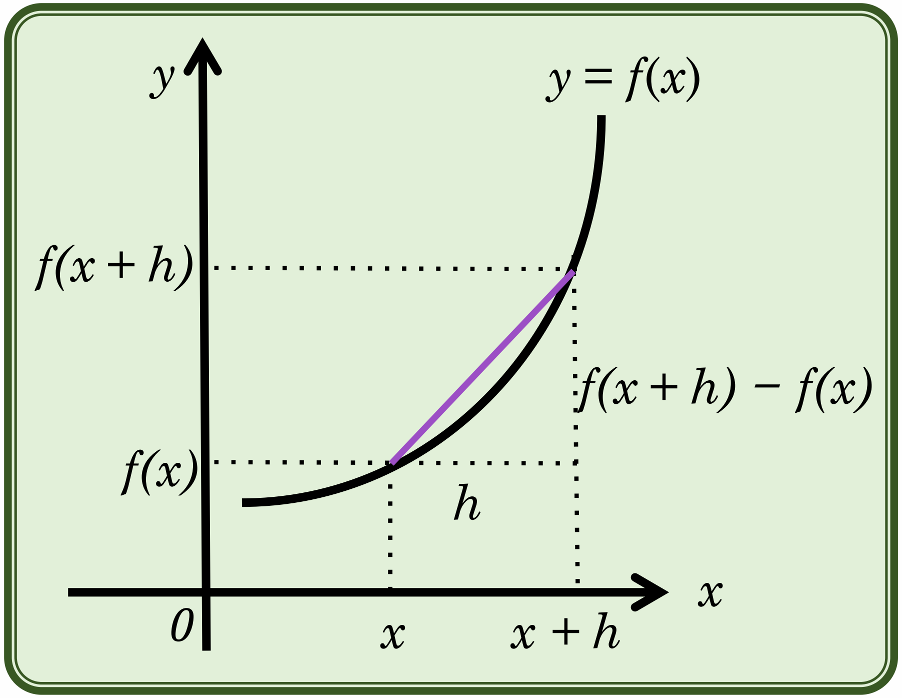
Step 4 Find the limit

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The Geometrical Meaning of Differentiation

If $y = f(x)$*, then* $\dfrac{dy}{dx} = f'(x) = \lim_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]$ *when the limit is exist. If the limit exist, function f is said to be differentiable with respect to x. The process of finding the differential coefficient of the function is called* ***differentiation.*** *The following notation is sometimes used and known as the* ***first principle****.*

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$



The Geometrical Meaning of Differentiation

Example 1

By using differentiation from the First Principle, find the derivatives of the following

a) $y = 9$

b) $y = x^2 + 1$

c) $y = \sqrt{x + 1}$

d) $y = \frac{1}{x}$

e) $y = 4x + 2x^2$

f) $y = \frac{1}{\sqrt{x - 1}}$

The Geometrical Meaning of Differentiation

a) Show that $f(x) = |x|$ is not differentiable at $x = 0$.

b) By using differentiation from the First Principle, find $f(x)$ if $f(x) = x^{1/3}$. Show that $f(x)$ is not differentiable at $x = 0$. Sketch its graph.

Example 2

The Geometrical Meaning of Differentiation

Theorem

The Relationship Between Differentiability and Continuity

If f(x) is differentiable at $x = x_0$, *then f(x) is continuous at* $x = x_0$

$$f'(x_0) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right]$$

The Geometrical Meaning of Differentiation

Find f'(a) for the given value of a from first principle.

a) $f(x) = 3x^2 - 12$ $\quad x_0 = 2$

b) $f(x) = \sqrt{x+1}$ $\quad x_0 = 3$

c) $f(x) = \dfrac{1}{\sqrt{x+1}}$ $\quad x_0 = 8$

Example 3

Differentiation of Simple Algebraic Functions

Differentiation of Simple Algebraic Functions

Theorem

The Derivative of a Constant Function

If $y = c$, where c is constant, then

$$\frac{dy}{dx} = 0$$

Differentiation of Simple Algebraic Functions

Example 4

Differentiate the following functions with respect to x.

a) $y = 3$

c) $y = \pi$

b) $y = {}^{7}\!/\!_{8}$

d) $y = \sqrt{e}$

Differentiation of Simple Algebraic Functions

Theorem

The Power Rule for Positive Integers

If $y = x^n$ and n is a positive integer,

then

$$\frac{dy}{dx} = nx^{n-1}$$

Differentiation of Simple Algebraic Functions

Example 5

Differentiate the following functions with respect to x.

a) $y = x^{10}$

c) $y = x^{15}$

b) $y = x^{99}$

d) $y = x^{20}$

Differentiation Rules

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Differentiation Rules

Theorem

The Derivative Constant Multiple Rule

If $y = cu$, where u is a differentiable function of x and c is a constant, then

$$\frac{dy}{dx} = c\frac{du}{dx}$$

Differentiation Rules

Example 6

Differentiate the following functions with respect to x.

a) $y = 7x^6$

b) $y = 100x^2$

c) $y = \frac{1}{3}x^{15}$

d) $y = -2x^{10}$

Differentiation Rules

Theorem

The Derivative Sum Rule

Let u and v be differentiable functions with respect to x. If $y = u + v$, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Differentiation Rules

Differentiate the following functions with respect to x.

a) $y = x^6 + x + 3$

c) $y = x^{10} - 5x^7 + 4x^5$

b) $y = (2x^2 - x)^2$

d) $y = -2(x-3)^{10}$

Example 7

Differentiation Rules

Theorem

The Derivative Product Rule

Let u and v be differentiable functions with respect to x. If $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Differentiation Rules

Differentiate the following functions with respect to x.

a) $y = (x + 1)(x + 3)^4$

b) $y = (x^2 - x)^2(x + 4)^2$

c) $y = x^2(5x^4 + 4)^2$

d) $y = (1 - x)^2(x + 5)^3$

Example 8

Differentiation Rules

Theorem

The Derivative Quotient Rule

Let u and v be differentiable functions with respect to x. If $y = \frac{u}{v}$ and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Differentiation Rules

Example 9

Differentiate the following functions with respect to x.

a) $y = \dfrac{5x^3 + x^2}{x^4 + 2}$

b) $y = \dfrac{7 - 3x^2}{3 - x}$

c) $y = \dfrac{1}{x}$

d) $y = \dfrac{1}{x^3 - 2x + 5}$

Differentiation Rules

Theorem

The Power Rule for Integers

If $y = x^n$ and n is any integer, then

$$\frac{dy}{dx} = nx^{n-1}$$

Differentiation Rules

Example 10

Differentiate the following functions with respect to x.

a) $y = x^{-2}$

b) $y = \dfrac{1}{x^9}$

c) $y = (x+4)^{-3}$

d) $y = \dfrac{1}{(x^2-3)^9}$

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Differentiation Rules

Theorem

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x. In other words, if $y = f(g(x))$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Differentiation Rules

Differentiate the following functions with respect to x.

a) $y = (x^5 - 5)^2$

b) $y = (8x + 5)^{12}(x^3 + 7)^{13}$

c) $y = \left(\frac{x-5}{2x+1}\right)^3$

d) $y = \sqrt{\frac{1-2x}{1+2x}}$

Example 11

Differentiation Rules

Exercises

Find dy/dx for the following functions. Hence, find the value(s) of x when dy/dx = 0.

a) $y = (2x + 3)(3x - 2)$

b) $y = \left(\frac{16}{x} - x\right)^2$

c) $y = (7x - 2)^2$

Differentiation Rules

$f(x)$	$f'(x)$
$y = cu$	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u \pm v$	$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Higher Order Differentiation

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Higher Order Differentiation

Occasionally, it is useful to differentiate the derivative of a function. In this context, we shall refer to f' as the first derivative of f and to the derivative of f' as the second derivative of f. We could denote the second derivative by $(f')'$, but for simplicity we write f''. Other higher-order derivatives are defined and denoted by f'''. In general, for $n > 3$, the n^{th} derivative of f is denoted by $f^{(n)}$, for example, $f^{(4)}$ or $f^{(5)}$.

Higher Order Differentiation

First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	or	$\frac{d}{dx}[f(x)]$
Second derivative	y''	$f''(x)$	$\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d^2y}{dx^2}$	or	$\frac{d^2}{dx^2}[f(x)]$
Third derivative	y'''	$f'''(x)$	$\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]=\frac{d^3y}{dx^3}$	or	$\frac{d^3}{dx^3}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	or	$\vdots$
n-th	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	or	$\frac{d^n}{dx^n}[f(x)]$

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Higher Order Differentiation

a) If $f(x) = 4x^4 - 2x^3 + 3$, find its first five derivatives.

b) Find y' and y'' for the following:

i. $y = 7x^4 - 5x^3 - 2x$

ii. $y = (x^3 + 5)(3x + 2)$

iii. $y = \dfrac{x^2}{x-2}$

iv. $y = \dfrac{1}{(x+1)^2(x+2)}$

Example 12

Higher Order Differentiation

a) Show that $y = x^3 + 3x + 1$ satisfies the equation $y''' + xy'' - 2y' = 0$.

b) If $y = 5x^6 - 2x^4$, find $\left.\frac{d^2y}{dx^2}\right|_{x=1}$

c) Verify that $y = Ax + Bx^2$ (A and B are constants) satisfy the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Example 13

Differentiation of Trigonometric Functions

Differentiation of Trigonometric Functions

STANDARD DERIVATIVES

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Differentiation of Trigonometric Functions

GENERALISED FORM

If u(x) is a function of x

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{cosec} u) = -\operatorname{cosec} u \cot u \frac{du}{dx}$$

Differentiation of Trigonometric Functions

Example 14

Differentiate the following functions with respect to x.

a) $y = \sin(2x)$

b) $y = \cos(x+3)$

c) $y = \sin^3(3x+1)$

d) $y = \sin\sqrt{2x+3}$

e) $y = \cos\sqrt{x^2+3}$

f) $y = \cos^2\sqrt{x^3-1}$

g) $y = \tan(x^2-1)$

h) $y = \tan^2(x^2+2)$

Differentiation of Trigonometric Functions

Example 15

Differentiate the following functions with respect to x.

$a)\ y = 3\sin(6x^2 - 12x)$

$b)\ y = \sin(x^2)\cos x$

$c)\ y = 5x\,\mathrm{cosec}(4x)$

$d)\ y = \cot(5x + 6)$

$e)\ y = \cos^3\left(\frac{2}{2 - 3x^2}\right)$

Differentiation of Logarithmic Functions

Differentiation of Logarithmic Functions

If $f(x) = \log_a x$ *is continuous for* $x > 0$*, and* a *is any base, then*

$$\frac{dy}{dx} = \frac{1}{x}\log_a e \ , \quad x > 0$$

If $a = e$*, then* $\log_e x = \ln x$*. Therefore*

$$\frac{dy}{dx} = \frac{1}{x}, \quad x > 0$$

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Differentiation of Logarithmic Functions

STANDARD DERIVATIVES

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e\,, \;\; x > 0$$

$$\frac{d}{dx}|\ln x| = \frac{1}{x}, \qquad x > 0$$

If u(x) is a function of x

$$\frac{d}{dx}(\log_a u) = \frac{1}{u}\log_a e\frac{du}{dx}, \;\; u > 0$$

$$\frac{d}{dx}|\ln u| = \frac{1}{u}\frac{du}{dx}, \qquad u \neq 0$$

GENERALISED FORM

Differentiation of Logarithmic Functions

Example 16

Differentiate the following functions with respect to x.

a) $y = \ln(3x)$

b) $y = \ln(x^2 - 2)$

c) $y = \ln\sqrt{3x - 1}$

d) $y = \ln(\sin 2x)$

e) $y = \ln(x^3 + x)$

f) $y = \ln(2\cos^5 x)$

Differentiation of Exponential Functions

Differentiation of Exponential Functions

If $y = a^x$, use the following steps:

Step 1: *Take logarithm with base e for both sides of the expression*

$$\ln y = \ln a^x$$
$$= x \ln a$$

Step 2: *Differentiate the expression with respect to x*

$$\frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = a^x \ln a$$

Differentiation of Exponential Functions

If $y = e^x$, use the following steps:

$$\frac{dy}{dx} = e^x$$

Differentiation of Exponential Functions

STANDARD DERIVATIVES

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}[e^x] = e^x$$

If u(x) is a function of x

GENERALISED FORM

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

Differentiation of Exponential Functions

Example 17

Differentiate the following functions with respect to x.

a)	$y = 5^x$	f)	$y = e^{x+1}$
b)	$y = 2^{x+5}$	g)	$y = e^{\sin(2x+3)}$
c)	$y = 3^{(2x-1)}$	h)	$y = e^{2x} - e^{-2x}$
d)	$y = 10^{\sin(3x+5)}$	i)	$y = e^{\sec x + \tan x}$

Differentiation of Exponential Functions

Example 18

1) If $y = e^x \sin 3x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

1) Find the derivatives of the following expressions.

a) $\frac{e^{3x}}{(x+1)^2}$

b) $(4x-1)e^{-2x^2}$

Differentiation of Implicit Functions

Differentiation of Implicit Functions

Theorem

If y and x are related in a function, and explicitly and implicitly defined, then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

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Differentiation of Implicit Functions

Example 19

1) Find the derivatives for $3x^2 - xy + 3y = 7$

a) by writing y in terms of x.

b) by using implicit differentiation.

2) Find the first derivatives for

a) $3y^2 - 2x^2 = 2xy$

b) $x^2 \sin y + 2x = y^2$

Differentiation of Implicit Functions

Example 20

Find $\frac{dy}{dx}$ for the following functions.

a) $y = \frac{x^2(x+2)}{x-3}$

b) $x^3 - xy + y^2 = 7$

c) $x^2y + e^{2x}y^2 - 2x = 0$

d) $y = \sin(x+y)^2$

e) $y = \frac{3x}{\sqrt{(x+1)(x+2)}}$

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Differentiation of Implicit Functions

1) If $y = \dfrac{\sqrt[3]{x+1}}{(x+2)\sqrt{x+3}}$, find $\dfrac{dy}{dx}$.

2) If $y = x^{-\frac{1}{2}} \cos x$, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0$$

Example 21

Differentiation of Parametric Functions

Differentiation of Parametric Functions

Theorem

$$\text{If } y = f(t) \text{ and } x = g(t) \text{ then}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$= f'(t) \times \frac{1}{g'(t)}$$

$$\frac{dy}{dx} = \frac{f'(t)}{g'(t)}$$

Differentiation of Parametric Functions

1) Find $\dfrac{dy}{dx}$

a) $y = t^3 + t, \qquad x = t^2$

b) $y = 4 - 4t - 4t^2, \qquad x = 2t$

1) Evaluate $\dfrac{dy}{dx}$

a) $x = t^2 - 2t, \quad y = t^3 - 3t, \qquad t = 4$

b) $x = \dfrac{3t}{1+t^3}, \quad y = \dfrac{3t^2}{1+t^3}, \qquad t = 2$

Example 22

Differentiation of Parametric Functions

Example 23

1) *The parametric equations of a curve is given by* $x = e^t$ *and* $y = \sin t$.

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *in terms of* t.

2) *Given* $y = \frac{t^2+4}{t}, x = \frac{t-3}{t}$, *find*

a) $\frac{dy}{dx}$ if $t = 1$

b) $\frac{d^2y}{dx^2}$ if $t = 1$