CHAPTER 3
APPLICATION OF DIFFERENTIATION

3.1 Approximate Value and Error

3.2 Rates of Change

3.3 Gradient of Curve at a Point

3.4 Maximum and Minimum

3.5 Curve Sketching

Approximate Value and Error

Approximate Value and Error

If $y = f(x)$ is a differentiable function, the first derivative dy/dx or $f'(x)$ is given by

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Observed that $\frac{f(x+\delta x) - f(x)}{\delta x}$ *does not equal to $f'(x)$, but when δx is very small, the value* $\frac{f(x+\delta x) - f(x)}{\delta x}$ *is an approximation of $f'(x)$.*

Approximate Value and Error

This can be written as

$$\frac{f(x+\delta x)-f(x)}{\delta x} \approx f'(x)$$

$$f(x+\delta x)-f(x) \approx f'(x)\delta x \qquad (1)$$

$$\delta f(x) \approx f'(x)\delta x \qquad (2)$$

where

$$\delta f(x) = f(x+\delta x)-f(x)$$

Approximate Value and Error

Formula (1) *and* (2) *can be used to find the approximate value of* $f(x + \delta x)$ *by using the exact value of* $f(x)$, $f'(x)$ *and* δx. *A small increment of* $f(x)$ *is obtained from a small increment of* x.

Approximate Value and Error

Example 1

1) *Find the approximate value of*

$a)\ \sqrt[3]{8.03}$ $b)\ \sqrt[3]{63}$

$c)\ \sqrt[4]{82.6}$

1) *The radius of a circle is measured with a possible error 2%. Find the possible percentage error in calculating the area of a circle*

Ans: 2.0025, 3.979, 3.015; 4%

Approximate Value and Error

Example 1

3) A closed cylinder has a height of 16 cm, radius r cm and the sum of surface area A cm^2. Prove that

$$\frac{dA}{dr} = 4\pi(r+8)$$

Find the approximate area if the radius increases from 4 cm to 4.02 cm.

4) Show that $x = 2$ can be chosen as an initial approximation of the equation $x^3 - 2x - 5 = 0$. Hence, find a better approximation.

$Ans: \frac{24}{25}\pi, \quad 2.1$

The Geometrical Meaning of Differentiation

Example 1

5) Given $V = \frac{4}{3}\pi r^3$ and r is decreasing from 16 to 15.8. Use differentiation to determine the decrement in V.

6) An inverted circular cone has a radius of 10 cm and a height of 15cm. When water level in the cone is x cm, the volume of the water is V cm3. Show that

$$V = \frac{4}{27}\pi x^3$$

If the water is flowing out of cone, find the approximate change in V, when x is decreasing from 4 cm to 3.95 cm.

$Ans: 204.8\pi; \quad -0.356\pi \text{ cm}^3$

Rates of Change

Rates of Change

If y is a function of x, then dy/dx is the rate of change of y with respect to x. In application, rates of change must be accompanied by appropriate units. In general, the units for a rate of change of y with respect to x are obtained by dividing the units of y by the units of x. Examples:

- *If y represents a displacement in meters (m) and t represents time in seconds (s), then dy/dt is the rate of change in y with respect to time t with units of meters per second (ms^{-1}).*
- *If p represents a pressure in atmosphere (atm) and V represents volume in liter (L), the dp/dV is the rate of change in p with respect to volume V with units of atmospheres per liter (atm L^{-1})*

Rates of Change

Example 2

1) The area of an ink blot at time t is $A\ cm^2$, where $A = 3t^2 + t$. Determine the rate of change of the blot area when $t = 5$.

2) The radius, r cm of a spherical balloon at time t seconds is given by

$$r = 3 + \frac{1}{1+t}$$

a) What is the initial radius of the sphere?

b) Determine the rate of change in the radius when $t = 2$. Is the radius decreasing or increasing?

$Ans: 31cms^{-1}:\ 4cm;\ -{}^{1}/_{9}\,cms^{-1}, decreasing$

Rates of Change

Example 2

3) *The length of the rope, l cm at t seconds is given by*

$$l = \frac{1}{4}t^4 - 4t + 10$$

Determine the time t when

a) *the length of the rope increases at the rate of change 4 cms^{-1}.*

b) *the length of the rope decreases at the rate of change 3 cms^{-1}.*

Ans: $2\ s$; $1\ s$

Rates of Change

Example 2

4) The radius, r cm of a circle at time t is given by $r = t^2 + 3$. Form an equation that relates the area, A cm^2 to t. Determine the rate of change of the area when $t = 1$.

(Give your answer in terms of π).

Ans: $A = \pi(t^4 + 6t^2 + 9)$; $16\pi\ cm^2 s^{-1}$

Constant Rate of Change

Let r cm be the radius of a circle at t seconds given by $r = 12 - 2t$, where $0 \leq t \leq 6$. Thus the rate of change of r with respect to t is $dr/dt = -2$ (a constant value), also known as a constant rate of change. It means that for every value of t, $0 \leq t \leq 6$, dr/dt is always the same. Hence, the constant rate of change is given by

$$\frac{dr}{dt} = \frac{\text{changing value of } r}{\text{changing value of } t}$$

Constant Rate of Change

Example 3

1) *The radius, r cm of a circle increases at a constant rate of* $0.5\ cms^{-1}$. *If the initial radius is* 3.5 *cm, find the radius of the circle after* 10 *seconds.*

Ans: $8.5cm$

Related Rates

The problem involving the rate of change of several related quantities is called the related rate of change problem. In general if y is a function of x, the problems involving the rates of change of y and x can be solved by using the chain rule

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

Related Rates

Example 4

1) Given that $y = (2x + 7)^{½}$ and y is increasing at a rate of 5 unit s^{-1}. Find the rate of change in x when

a) $x = 0.3$ b) $y = 3$

2) The radius of a circle is increasing at the rate of 5 cm per minute. Find

a) the rate of change of the area of the circle when its radius is 12 cm.

b) the radius of the circle when its area is increasing at a rate of 50π cm^2 per minute.

Ans: $13.784\ unit\ s^{-1}$; $15\ unit\ s^{-1}$: $120\pi cm^2\ min^{-1}$; $5\ cm$

Related Rates

Example 4

3) *A hemispherical bowl has a radius of 10 cm and the depth of water in the bowl is h cm. The formula of the volume of water is* $V = \frac{1}{3}\pi h^2(30 - h)$. *Water is poured into the bowl at the rate of* $2\ cm^3\ s^{-1}$. [*Volume hemispherical bowl,* $V = \frac{2}{3}\pi r^3$].

a) *If the radius of the water surface is r cm, state r in terms of h.*

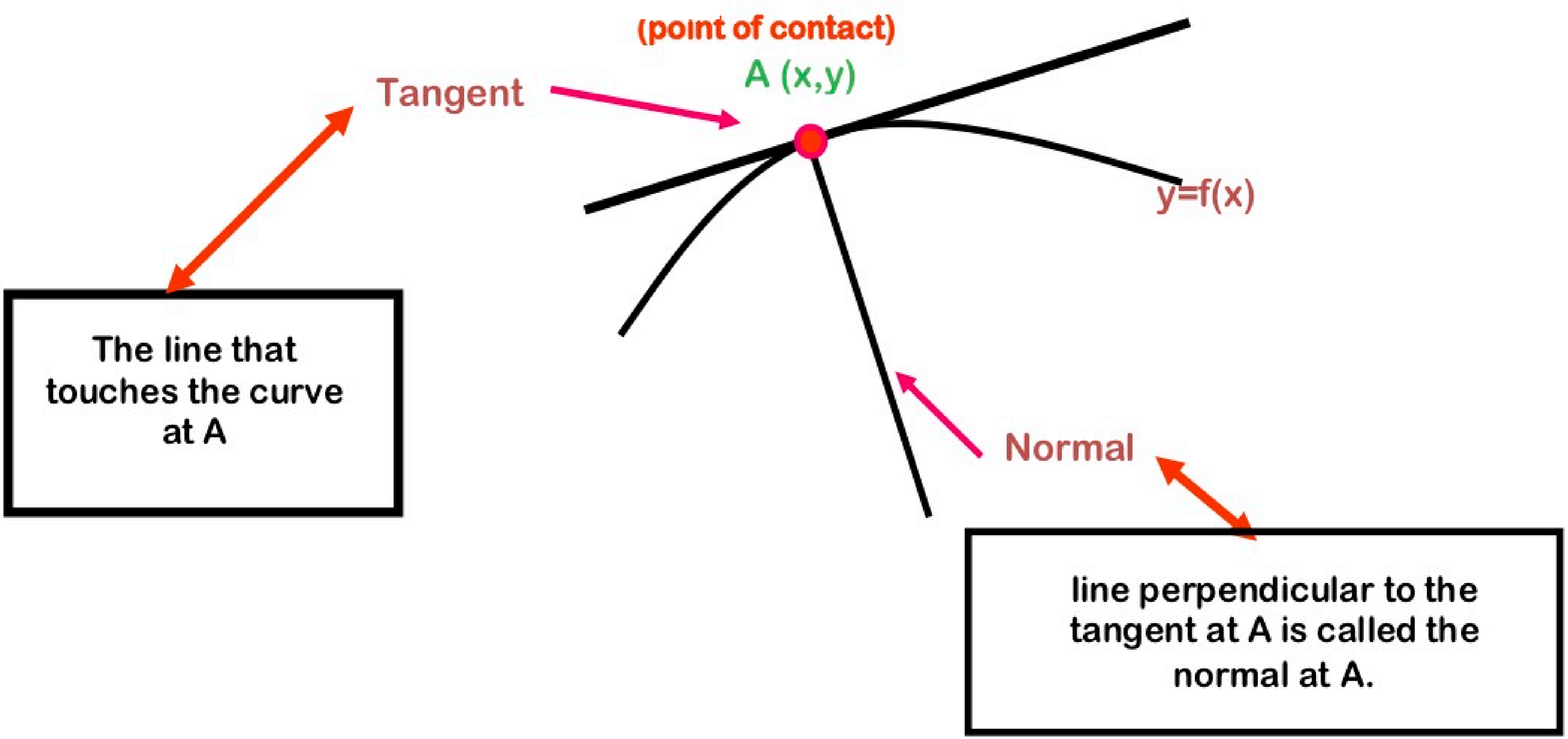
b) *When h = 4 cm, determine the rates of change in the height and the radius of the water surface.*

$Ans:\ r = \sqrt{h(20-h)}\ ;\ \frac{dh}{dt} = \frac{1}{32\pi}\ cm\ s^{-1},\ \frac{dr}{dt} = \frac{3}{128\pi}\ cm\ s^{-1}$

Gradient of Curve At A Point

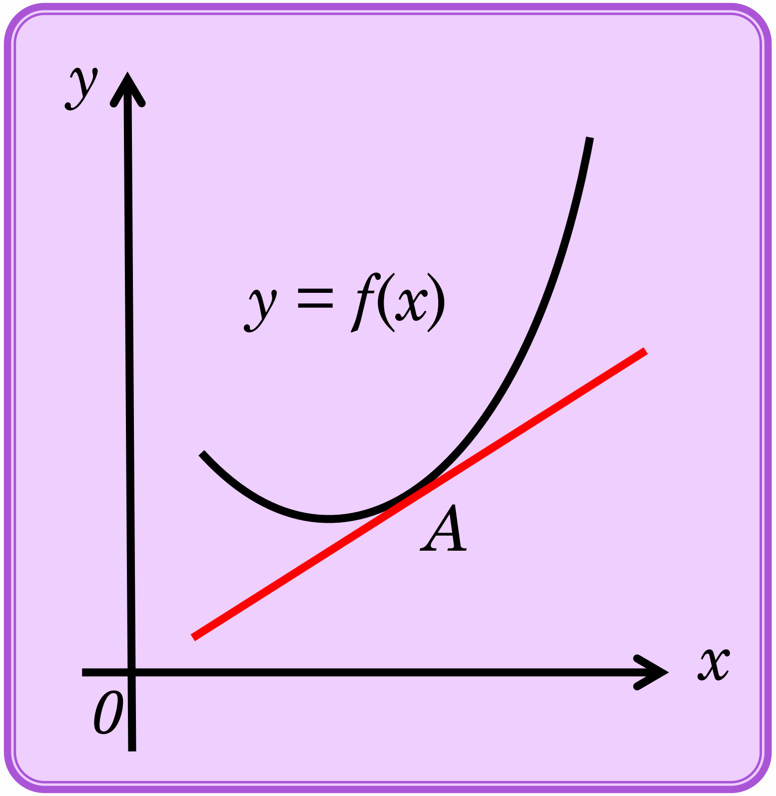
Gradient of Curve at A Point

Tangent and Normal Lines



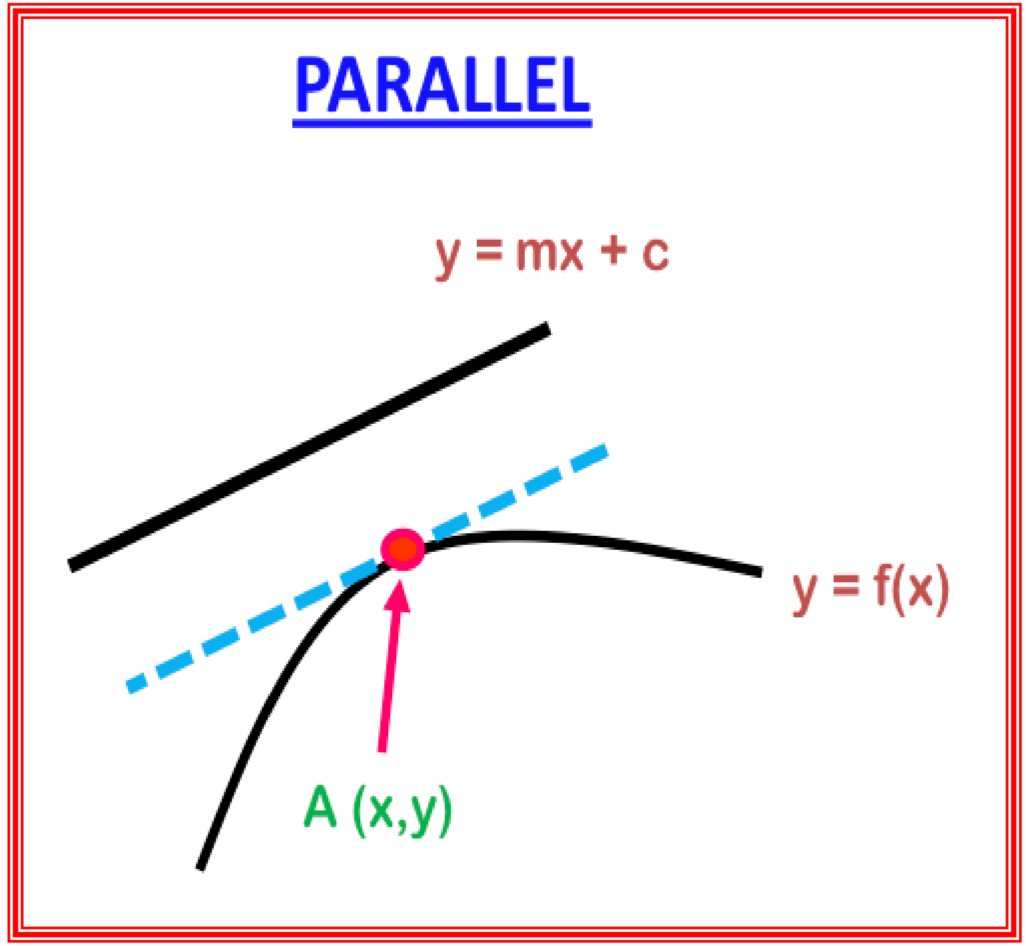
Gradient of Curve at A Point

The gradient of curve at a point can be defined as the gradient of tangent line at the point of the curve. Let the straight line (red line) be a tangent to the curve at A, where A is a point moving along the curve $y = f(x)$. The gradient of the tangent to the curve changes according to the point on the curve. These gradients can be obtained by substituting the coordinate points in dy/dx or $f'(x)$.



Equation of Tangent to a Curve

The tangent to the curve $y = f(x)$ at any point A can be defined as a straight line (dotted blue line) which touches the curve at point A. To find the equation of the tangent to the curve at A, we need to find the gradient of the tangent to curve at that point.



Equation of Tangent to a Curve

Gradient or Slope

Given m_1 *and* m_2 *are the gradients of Line 1 and Line 2 respectively.*

The lines are parallel if $m_1 = m_2$

The gradient of the curve at A is the same as the gradient of the tangent to the curve at A.

Equation of Tangent to a Curve

Gradient or Slope

At the point $A(x, y)$ on the curve $y = f(x)$, the equation of the **TANGENT** is

$$y - y_1 = m_T(x - x_1)$$

where

$$m_T = \frac{dy}{dx}$$

Equation of Tangent to a Curve

a) *Find the gradient of the curve* $y = x^3 + 8x - 5$ *at* $(0, -5)$ *and* $(2, 19)$.

b) *Find the gradient of the tangent to the following curve at the given point.*

i. $y = x^2 - 4x + 4$ *at the point where* $y = 1$

ii. $y = x^3$ *at the point* $(2,8)$

iii. $y = x^3 + 2x^2 - x$ *at the point where* $x = -2$

Example 5

$Ans:\ 8, 20\ ;\ -2, 2{:}\ 12{:}\ 3$

Equation of Tangent to a Curve

Example 5

c) Find the equation of a tangent to the parabola $y = x^2 - 5x + 3$ at $x = 3$.

d) Find the the equation of a tangent to the ellipse $3x^2 + 4y^2 = 48$ at $(2, 3)$.

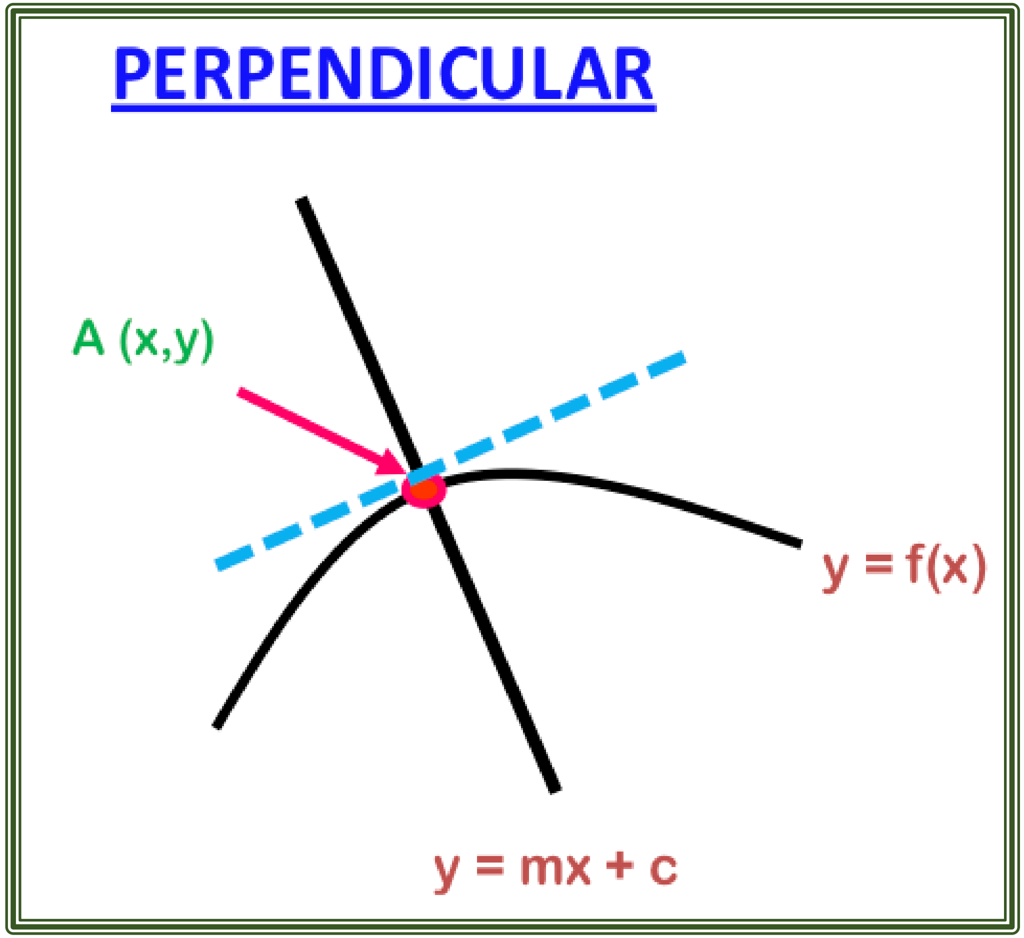
e) Find the equation of the tangent for the curve $y = x^2 + 2x + 3$ which parallel to the line $y = 4x + 1$.

f) The gradient of the curve $y = ax^2 + bx$ at $(1, 3)$ is 4. Find the values of a and b.

Ans: $y = x - 6$; $2y + x = 8$; $y = 4x + 2$; $a = 1, b = 2$

Equation of Normal to a Curve

The normal line equation to the curve y = f(x) at any point A can be defined as a straight line (straight line) that is perpendicular to the tangent (dotted line). If the gradient of the tangent to the curve y = f(x) at A is m, then the gradient of the normal at A is −1/m.



Equation of Normal to a Curve

Gradient of the Normal

Given m_1 and m_2 are the gradients of Line 1 and Line 2 respectively.

The lines are perpendicular if $m_1 \cdot m_2 = -1$.

Notes:

gradient of the tangent × **gradient of the normal** = −1

If the gradient of the tangent is m, then the gradient of the normal line is $-1/m$.

Equation of Normal to a Curve

Gradient of the Normal

At the point $A(x, y)$ on the curve $y = f(x)$, the equation of the ***NORMAL*** *is*

$$y - y_1 = m_N(x - x_1)$$

where

$$m_N = -\frac{1}{m_T}$$

Equation of Normal to a Curve

Example 6

a) *Find the normal of the curve* $y = x^3 + 8x - 5$ *at* $(0, -5)$ *and* $(2, 19)$.

b) *Find the normal to the following curve at the given point.*

i. $y = x^2 - 4x + 4$ *at the point where* $y = 1$

ii. $y = x^3$ *at the point* $(2,8)$

iii. $y = x^3 + 2x^2 - x$ *at the point where* $x = -2$

Ans: $-\frac{1}{8}, -\frac{1}{20};\ -\frac{1}{2}, \frac{1}{2};\ -\frac{1}{12};\ -\frac{1}{3}$

Equation of Normal to a Curve

Example 6

c) *Find the equation of a normal to the parabola* $y = x^2 - 5x + 3$ *at* $x = 3$.

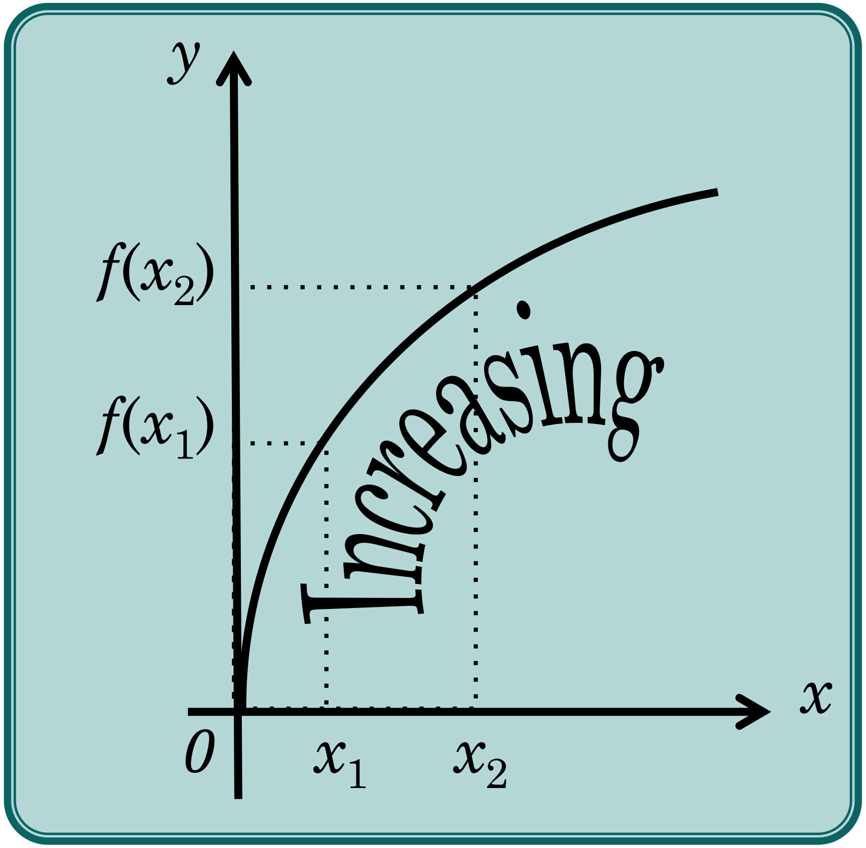
d) *Find the the equation of a normal to the ellipse* $3x^2 + 4y^2 = 48$ *at* $(2, 3)$.

e) *Find the equation of the normal to the curve* $y = x^2 - 5x + 1$ *which perpendicular to the line* $y = x - 7$

Ans: $x + y = 0$; $y = 2x - 1$; $x + y + 2 = 0$

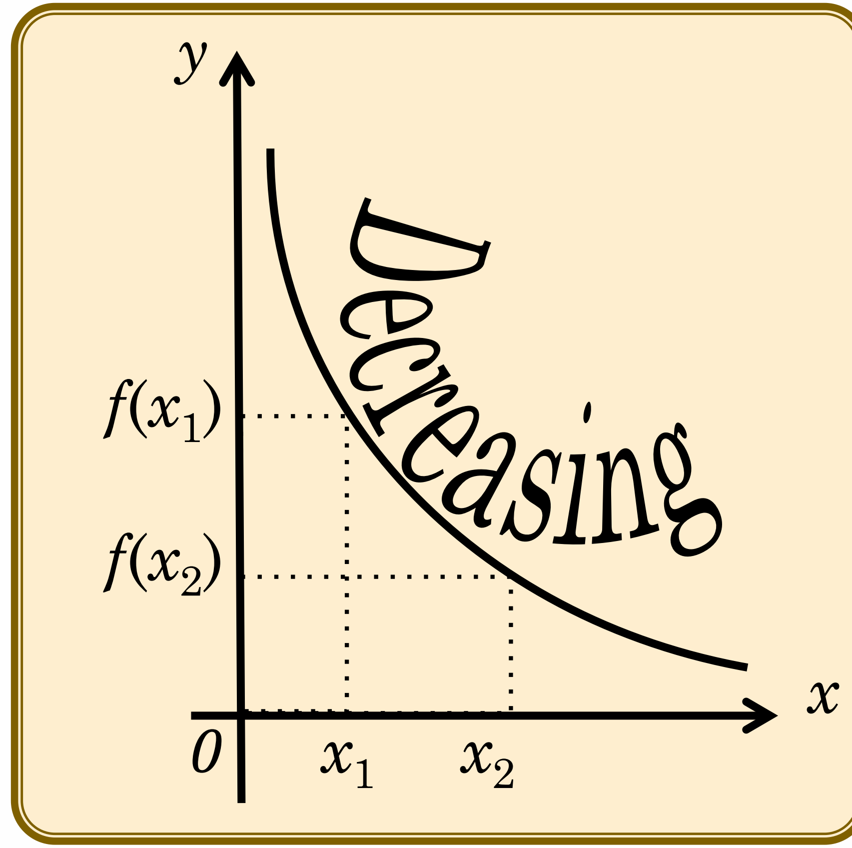
Maximum and Minimum

Maximum and Minimum



$f(x_1) < f(x_2)$

Increasing function



$f(x_1) > f(x_2)$

Decreasing function

Maximum and Minimum

Theorem

Critical Point and Critical Number

A point (x_0, y_0) is a critical point and $x = x_0$ is its critical number of $f(x)$, if $f'(x) = 0$ of if $f'(x_0)$ does not exist for the interval $a < x_0 < b$.

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Maximum and Minimum

Definition

Local Maximum and Minimum Values

The number $f(x_0)$ is a

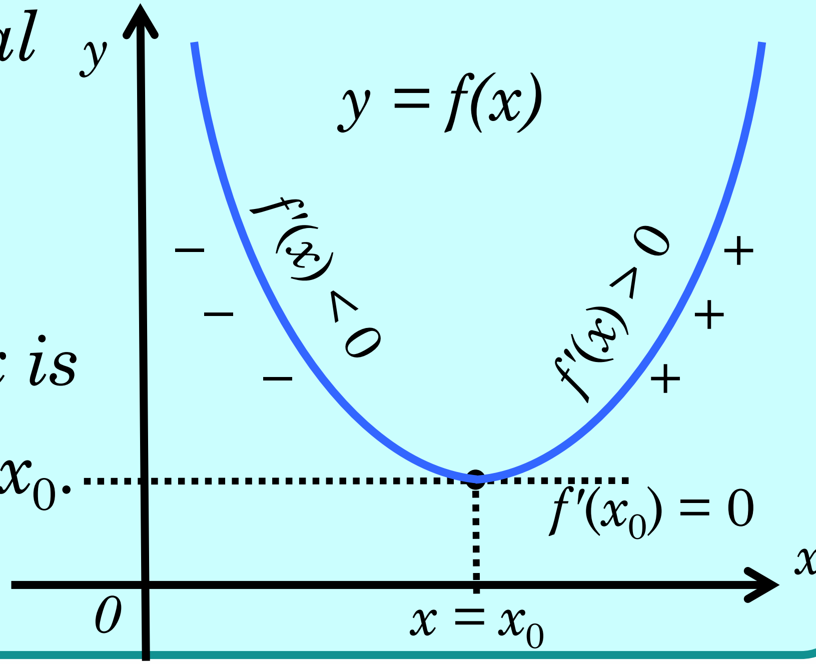
- local maximum value of $f(x)$ if $f(x_0) \geq f(x)$ when x is near x_0.
- local minimum value of $f(x)$ if $f(x_0) \leq f(x)$ when x is near x_0

Maximum and Minimum

First Derivative Test

Given that $y = f(x)$ is a continuous function

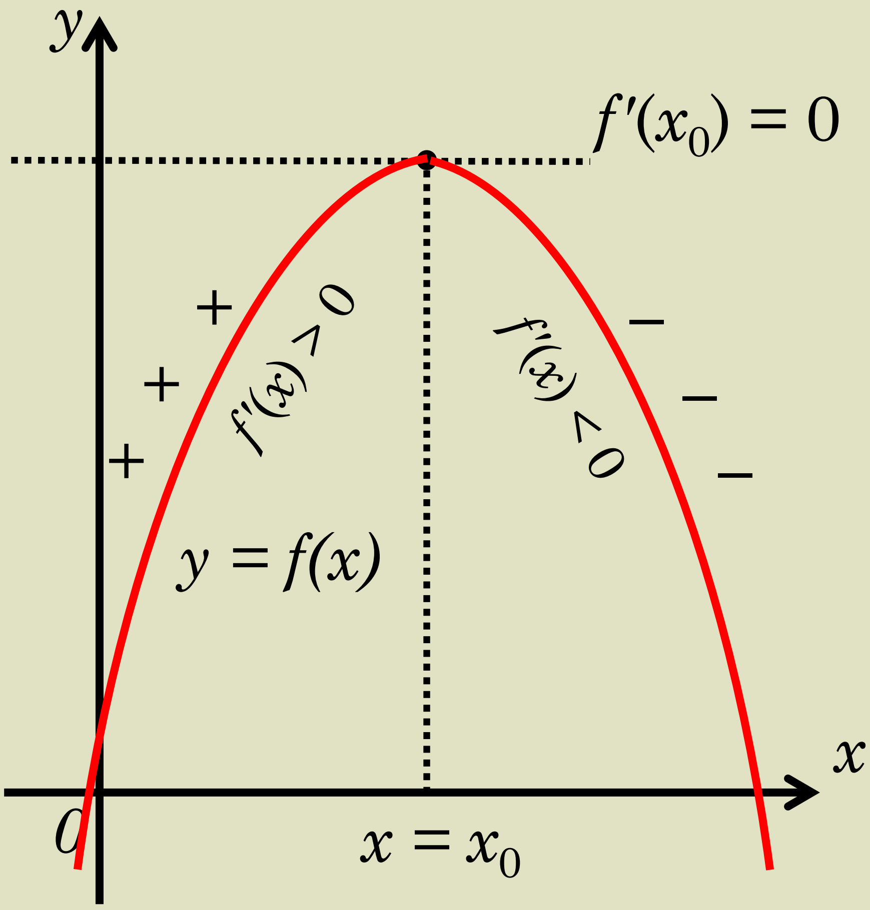
- If $f'(x_0) = 0$, or if $f'(x_0)$ does not exist (not defined), x_0 is a critical number.
- A critical point is a local minimum point if $f'(x)$ changes sign from negative to positive as x is increasing through $x = x_0$. The curve is concave.



Maximum and Minimum

First Derivative Test

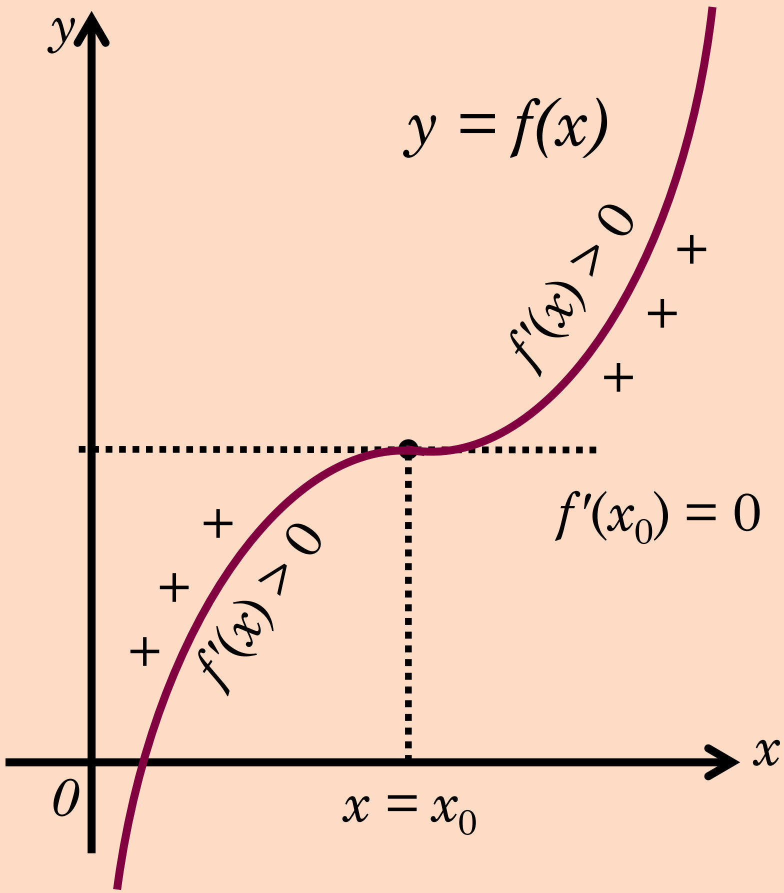
- *A critical point is a local maximum point if $f'(x)$ changes sign from positive to negative as x is increasing through $x = x_0$. The curve is convex.*



Maximum and Minimum

First Derivative Test

- *A critical point is neither a local maximum nor a local minimum if $f'(x)$ does not change sign as x is increasing through $x = x_0$.*



Maximum and Minimum Value in the Interval

Example 7

Find the local maximum and minimum points (if any) of the functions below:

a) $f(x) = x^2 - 4x - 1$

b) $f(x) = x^3 - 9x^2 + 15x - 5$

c) $f(x) = x^3 - 3x + 2$

Maximum and Minimum Value in the Interval

The local maximum and minimum points of

$f(x) = x^2 - 4x - 1 \quad \Rightarrow \quad f'(x) = 2x - 4 = 2(x - 2)$

Values of x	**1.5**	**2**	**2.5**
Signs of $f'(x)$	–	**0**	**+**
Slope Sketching	↘	→	↗
Shape of the graph	**Concave**		

Example 7

Ans: local minimum point $(2, -5)$

Maximum and Minimum Value in the Interval

Example 7

The local maximum and minimum points of

$f(x) = x^3 - 9x^2 + 15x - 5$

=> $f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$

Values of x	**0.5**	**1**	**1.5**
Signs of $f'(x)$	+	0	–
Slope Sketching	↗	→	↘
Shape of the graph	**Convex**		

Ans: local maximum point (1,2)

Maximum and Minimum Value in the Interval

Example 7

The local maximum and minimum points of

$f(x) = x^3 - 9x^2 + 15x - 5$

=> $f'(x) = 3x^2 - 18x + 15 = 3(x - 1)(x - 5)$

Values of x	**4.5**	**5**	**5.5**
Signs of $f'(x)$	–	**0**	**+**
Slope Sketching	↘	→	↗
Shape of the graph	**Concave**		

Ans: local minimum point $(5, -30)$

Maximum and Minimum Value in the Interval

Example 7

The local maximum and minimum points of

$f(x) = x^3 - 3x + 2$ => $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Values of x	**–1.5**	**–1**	**–0.5**
Signs of $f'(x)$	**+**	**0**	**–**
Slope Sketching	↗	→	↘
Shape of the graph	**Convex**		

Ans: local maximum point $(-1,4)$

Maximum and Minimum Value in the Interval

The local maximum and minimum points of

$f(x) = x^3 - 3x + 2$ => $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Values of x	**0.5**	**1**	**1.5**
Signs of f′(x)	–	**0**	**+**
Slope Sketching	↘	→	↗
Shape of the graph	***Concave***		

Example 7

Ans: local minimum point (1,0)

Maximum and Minimum

Second Derivative Test

Assuming that $y = f(x)$ *has a critical number* $x = x_0$

1. *If* $f''(x_0) < 0$, *the graph is convex and* $f(x)$ *has a local maximum value at* $x = x_0$.
2. *If* $f''(x_0) > 0$, *the graph is concave and* $f(x)$ *has a local minimum value at* $x = x_0$.
3. *If* $f''(x_0) = 0$, *or does not exist, the second derivative test fails. We have to see the first derivative test to determine the property of the extreme points at* $x = x_0$.

Maximum and Minimum

Example 8

Find the local maximum and minimum points (if any) of the functions below:

a) $f(x) = \frac{1}{3}x^3 - 3x^2 + 8x + 1$

b) $f(x) = 1 - x^4$

Maximum and Minimum

Example 8

The local maximum and minimum points of

$f(x) = \frac{1}{3}x^3 - 3x^2 + 8x + 1$

$\Rightarrow f'(x) = x^2 - 6x + 8 = (x-2)(x-4)$

$\Rightarrow f''(x) = 2x - 6 = 2(x-3)$

Values of x	**1.5**	**2**	**2.5**
Signs of $f'(x)$	**+**	**0**	–
Slope Sketching	➚	➔	➘
Signs of $f''(x)$	–	–	–
Shape of the graph	**Convex**		

Ans: $f''(x) < 0$; local maximum point $\left(2, 7\frac{2}{3}\right)$

Maximum and Minimum

The local maximum and minimum points of

$f(x) = \frac{1}{3}x^3 - 3x^2 + 8x + 1$

$\Rightarrow f'(x) = x^2 - 6x + 8 = (x - 2)(x - 4)$

$\Rightarrow f''(x) = 2x - 6 = 2(x - 3)$

Values of x	3.5	4	4.5
Signs of $f'(x)$	–	0	+
Slope Sketching	↘	→	↗
Signs of $f''(x)$	+	+	+
Shape of the graph	Concave		

Example 8

Ans: $f''(x) > 0$; local minimum point $\left(4, 6\frac{1}{3}\right)$

Maximum and Minimum

Example 8

The local maximum and minimum points of

$f(x) = 1 - x^4$

$\Rightarrow f'(x) = -4x^3$

$\Rightarrow f''(x) = -12x^2$

Values of x	-0.5	0	0.5
Signs of $f'(x)$	+	0	–
Slope Sketching	↗	→	↘
Signs of $f''(x)$	–	0	–
Shape of the graph	Convex		

$Ans: \ f''(x) = 0; \ local\ maximum\ point\ (0, 1)$

Maximum and Minimum

Definition

Point of Inflection

A point which separates a convex and a concave sections of a continuous function is called a ***point of inflection****.*

Maximum and Minimum

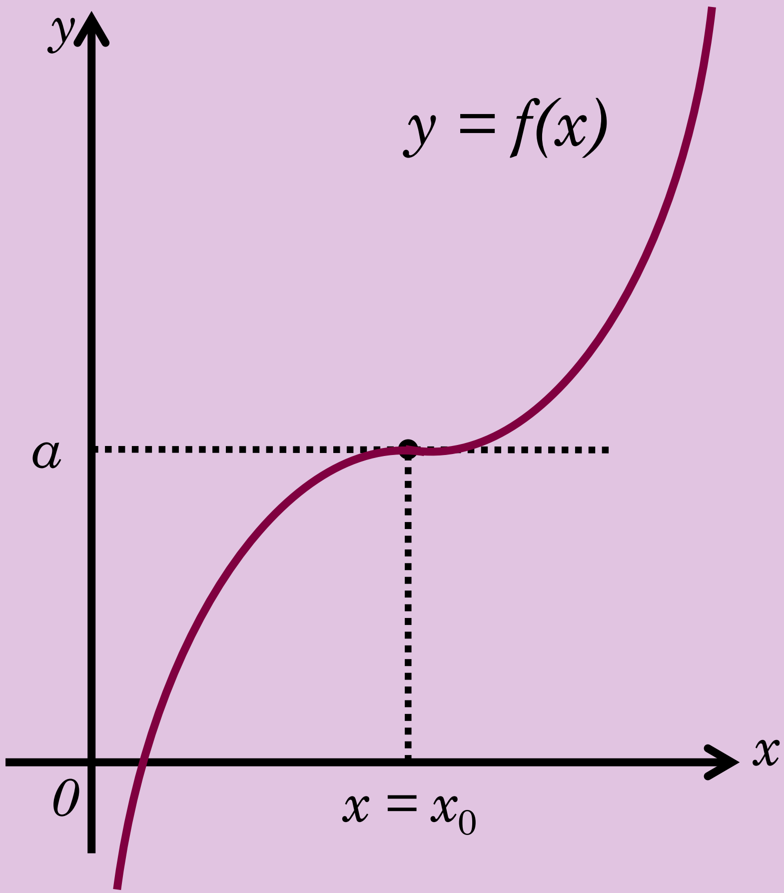
Point of Inflection

- $f(x_0) = 0.$

  $f'(x_0) = 0.$

  $f''(x_0) = 0.$

  Thus, (x_0, a) is a point of inflection and it is also a critical point.



Maximum and Minimum

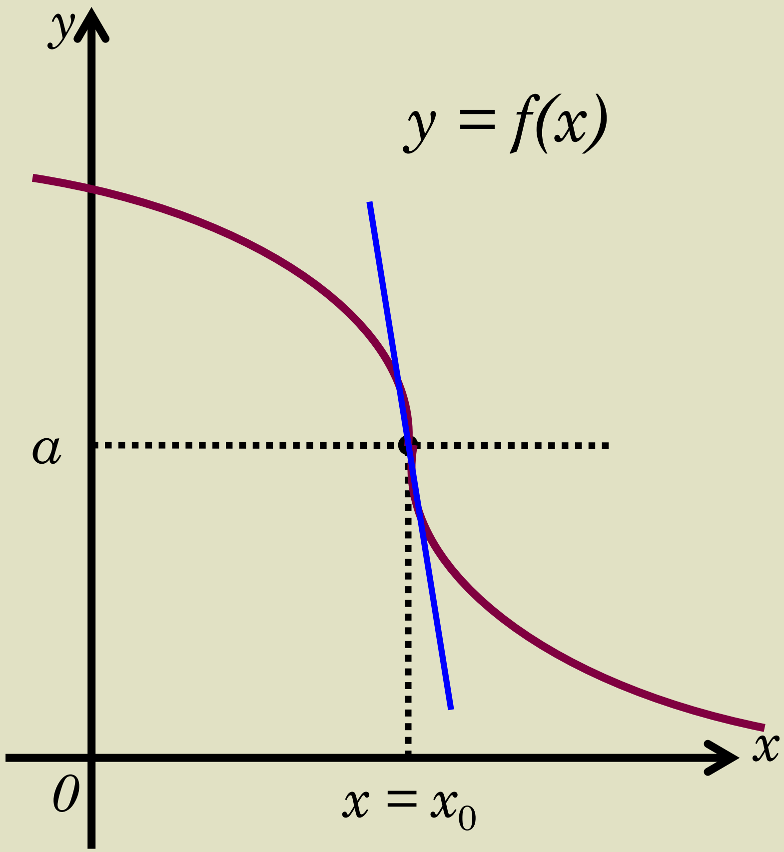
Point of Inflection

- $f(x_0) = a.$

$f'(x_0) \neq 0.$

$f''(x_0) = 0.$

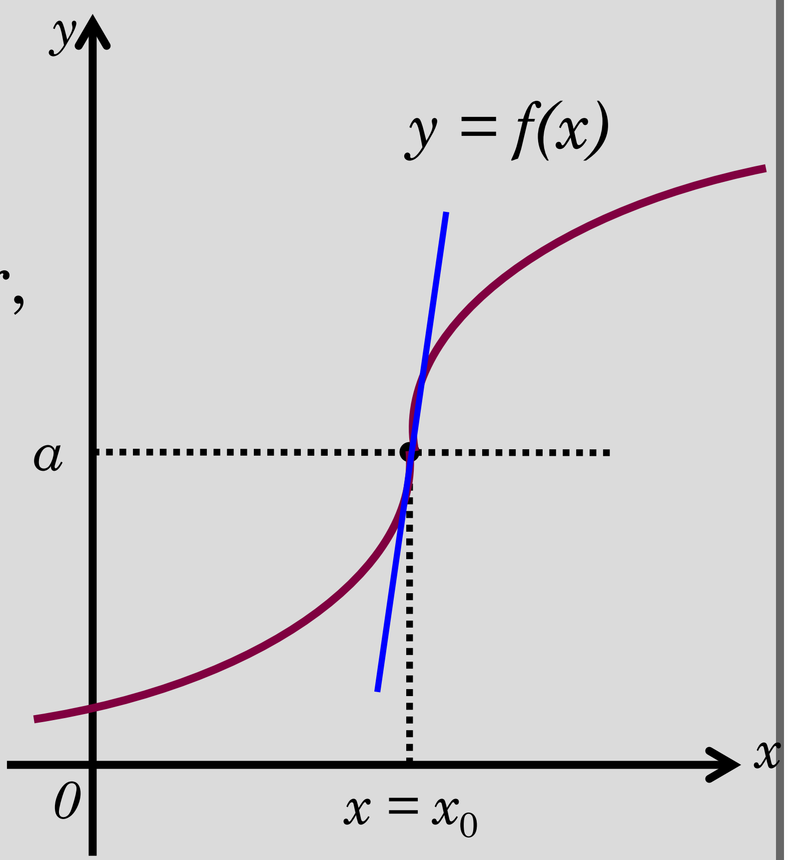
Since $f'(x_0) \neq 0$, therefore (x_0, a) is a point of inflection but not a critical point.



Maximum and Minimum

Point of Inflection

- $f(x_0) = a$.
  $f'(x_0)$ does not exist.
  $f''(x_0)$ does not exist.
  $x = x_0$ is a critical number,
  Since the curves changes from being concave to convex at $x = x_0$, therefore (x_0, a) is a point of inflection. However, it is neither a local maximum nor a local minimum.



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Maximum and Minimum

Theorem

Point of Inflection Sign $f'(x_o)$

Let $y = f(x)$ be a continuous function. If $f''(x_0) = 0$ or if $f''(x_0)$ does not exist and if the value of $f''(x)$ changes sign when passing through $x = x_0$, then the point $(x_0, f(x_0))$ on the curve is a ***point of inflection.***

Maximum and Minimum

Example 9

Determine the local maximum and minimum points of the function below:

a) $f(x) = (x - 1)^3$

b) $f(x) = x^4 - 4x^3$

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Maximum and Minimum

Example 9

The local maximum and minimum points of

$f(x) = (x - 1)^3$

$\Rightarrow f'(x) = 3(x - 1)^2$ → $f'(x) = 0$ for critical point

$\Rightarrow f''(x) = 6(x - 1)$ → $f''(1) = 0$ for inflection point

Values of x	**0.5**	**1**	**1.5**
Signs of $f'(x)$	+	0	+
Slope Sketching	↗	→	↗
Signs of $f''(x)$	–	0	+
Shape of the graph	**Convex**		**Concave**

Ans: inflection point $(1,0)$

Maximum and Minimum

Example 9

The local maximum and minimum points of

$f(x) = x^4 - 4x^3$

$\Rightarrow f'(x) = 4x^3 - 12x^2$ → $f'(x) = 0$ for critical point

$\Rightarrow f''(x) = 12x^2 - 24x$ → $f''(x) = 0$ for inflection point

Values of x	**-0.5**	**0**	**0.5**
Signs of $f'(x)$	–	**0**	–
Slope Sketching	↘	→	↘
Signs of $f''(x)$	+	**0**	–
Shape of the graph	**Concave**		**Convex**

Maximum and Minimum

Example 9

The local maximum and minimum points of

$f(x) = x^4 - 4x^3$

$\Rightarrow f'(x) = 4x^3 - 12x^2$ → $f'(x) = 0$ for critical point

$\Rightarrow f''(x) = 12x^2 - 24x$ → $f''(x) = 0$ for inflection point

Values of x	**2.5**	**3**	**3.5**
Signs of $f'(x)$	–	**0**	**+**
Slope Sketching	↘	→	↗
Signs of $f''(x)$	+	+	+
Shape of the graph	**Concave**		**Concave**

Maximum and Minimum

Example 9

$f(x) = x^4 - 4x^3$

$\Rightarrow f'(x) = 4x^3 - 12x^2 \rightarrow f'(x) = 0$ *for critical point*

$\Rightarrow f''(x) = 12x^2 - 24x \rightarrow f''(x) = 0$ *for inflection point*

Values of x	−0.5	0	0.5	1.5	2	2.5	3	3.5
Signs of $f'(x)$	−	0	−	−	−	−	0	+
Slope Sketching	↘	→	↘	↘	↘	↘	→	↗
Signs of $f''(x)$	+	0	−	−	0	+	+	+
Shape of the graph	Concave		Convex	Convex		Concave		Concave

Ans: local minimum $(3, -27)$ *inflection points* $(0, 0)$ & $(2, -16)$

Maximum and Minimum

Definition

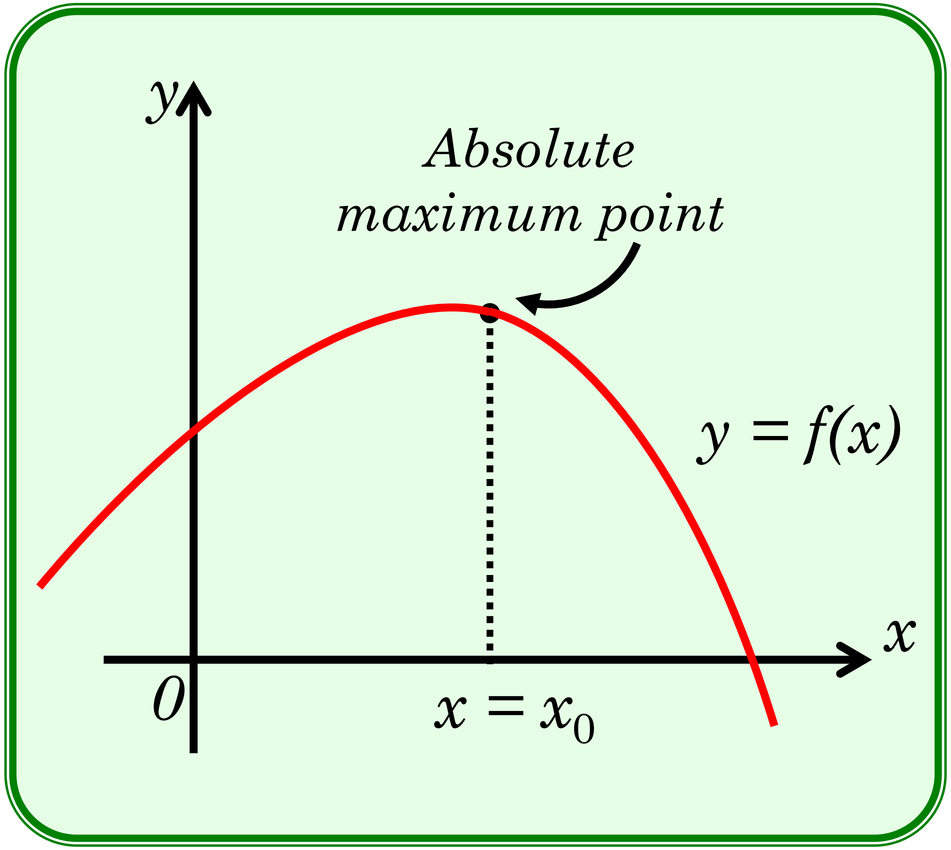
Absolute Maximum and Minimum Values

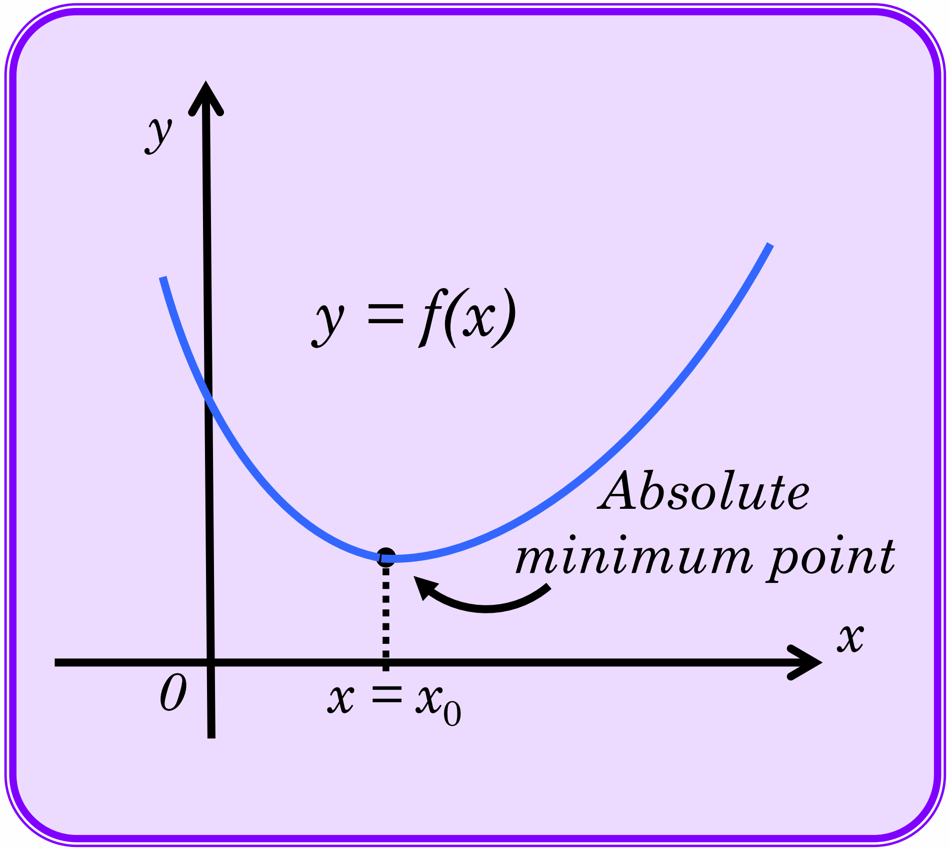
Let $x = x_0$ be a number in the domain D of a function $f(x)$. Then $f(x_0)$ is the

- *absolute maximum value of $f(x)$ if $f(x_0) \geq f(x)$ for all x in D.*
- *absolute minimum value of $f(x)$ if $f(x_0) \leq f(x)$ for all x in D.*

The maximum and minimum values of $f(x)$ are called ***extreme values*** *of $f(x)$.*

Maximum and Minimum





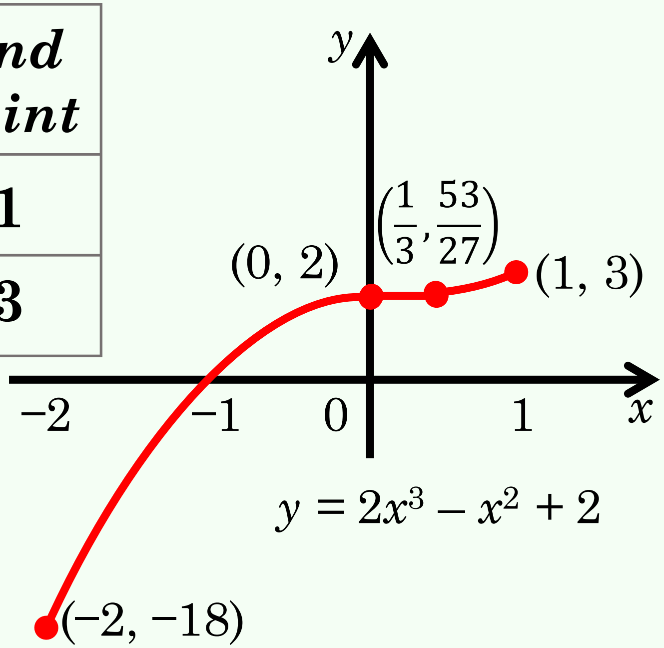
Maximum and Minimum

Example 10

Find the absolute maximum and minimum of the function $f(x) = 2x^3 - x^2 + 2$ for $-2 \le x \le 1$. *Hence, sketch the graph.*

$\Rightarrow f'(x) = 6x^2 - 2x = 2x(3x - 1)$ => *critical numbers*

Values of x	End point	Critical number	Critical number	End point
	–2	0	⅓	1
f(x)	–18			3



Absolute maximum of f(x) is _____

Absolute minimum of f(x) is _____

Ans: Absolute minimum value $= -18$; *absolute minimum value* $= 3$

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Maximum and Minimum

Example 11

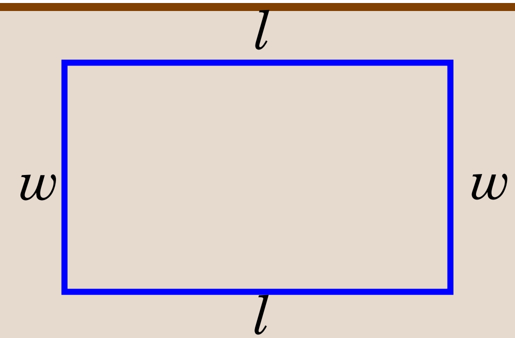
1. A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running meter of chicken wire is available for the fence?
2. An open box is to be made from a 16 cm by 30 cm piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?
3. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches.

Maximum and Minimum

Example 11

l = length of the rectangle (m)

w = width of the rectangle (m)



A = area of the rectangle (m^2) = lw (1)

Given that

perimeter of the rectangle = $100 = 2l + 2w$

$\Rightarrow w = 50 - l$ (2)

Substitute (2) in (1)

$\Rightarrow A = l(50 - l) = 50l - l^2$ => identify domain

Domain for length, l is ___________

Ans: $0 \leq l \leq 50$

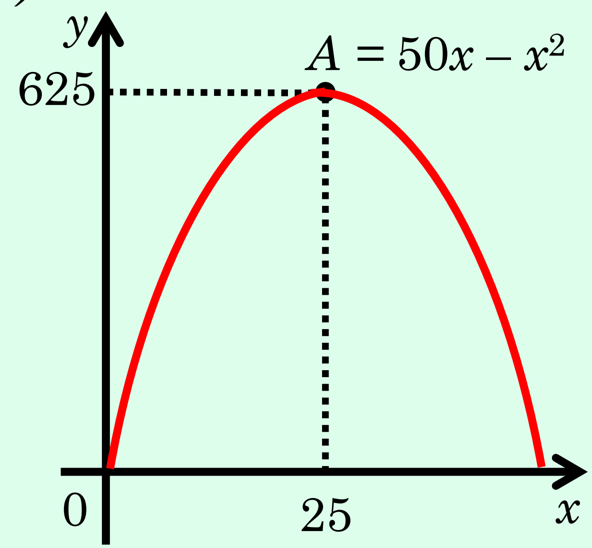
Maximum and Minimum

Example 11

Differentiate the area, A

$$\frac{dA}{dl} = 50 - 2l \quad \Rightarrow \text{(identify critical points)}$$

Values of x	End point	Critical number	End point
	0	25	50
Area, A	0		



Width, w = _____

The corresponding value of w is ___, so the rectangle of perimeter 100 meter with greatest area is a square with sides of length ___.

Maximum and Minimum

Example 11

x = *length (in cm) of the sides of the squares to be cut out*

V = *volume (in cubic cm) of the resulting box*

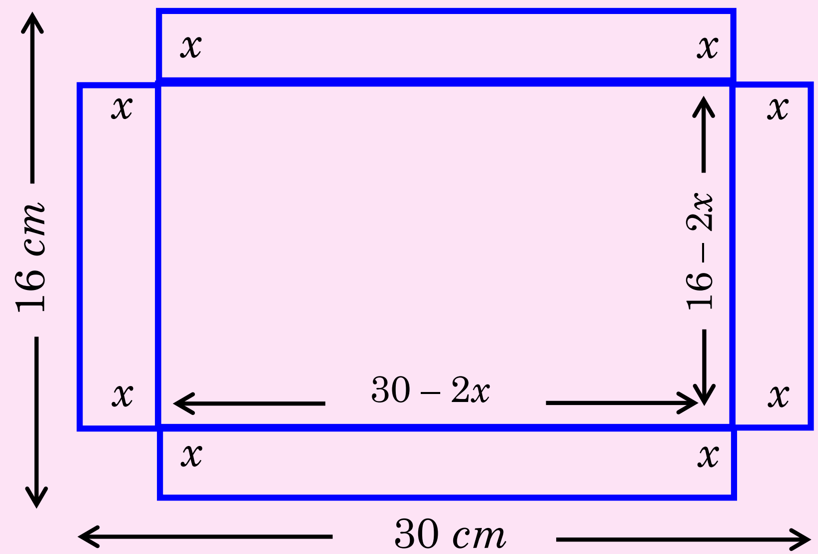
$= (16-2x)(30-2x)x$

$= 480x - 92x^2 + 4x^3 \quad (1)$

$\Rightarrow$ *identify domain*

Domain for length, x is

______



Ans: $0 \le x \le 8$

Maximum and Minimum

Differentiate the volume V

$$\frac{dV}{dr} = 480 - 184x + 12x^2 \quad => \textit{(identify critical points)}$$

$= 4(x - 12)(3x - 10)$

Values of x	Endpoint	Critical number	Endpoint
	0	10/3	8
Volume, V	0		

length, $x =$ _____

The greatest possible volume $V \approx 726\, cm^3$ *occurs when we cut out squares whose sides have length* 10/3 *cm.*

Example 11

Ans: $0 \le r \le 6$

Maximum and Minimum

r = radius (in inches) of the cylinder
h = height (in inches) of the cylinder
V = volume (in cubic inches) of the cylinder

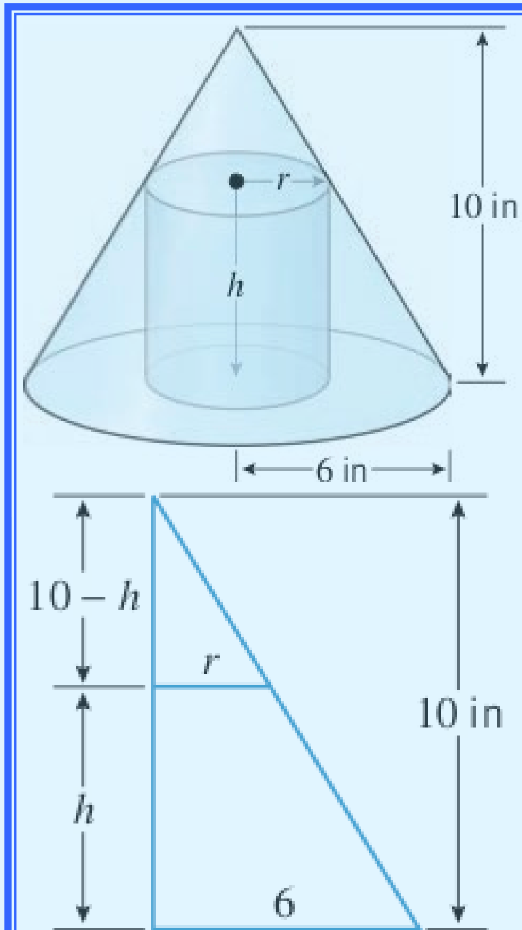
V = Volume of the inscribed cylinder
$= \pi r^2 h$ (1)

Using similar triangles

$$\frac{10-h}{r} = \frac{10}{6} \Rightarrow h = 10 - \frac{5}{3}r \qquad (2)$$

Substitute (2) in (1)

$$V = \pi r^2\left(10 - \frac{5}{3}r\right)$$ => identify domain



Example 11

Maximum and Minimum

Example 11

Domain for radius, r is __________

Differentiate the volume V

$$\frac{dV}{dr} = \pi(20r - 5r^2)$$ => (identify critical points)

Values of r	Endpoint	Critical number	Endpoint
	0	**4**	**6**
Volume, V	**0**		

height, h = _____

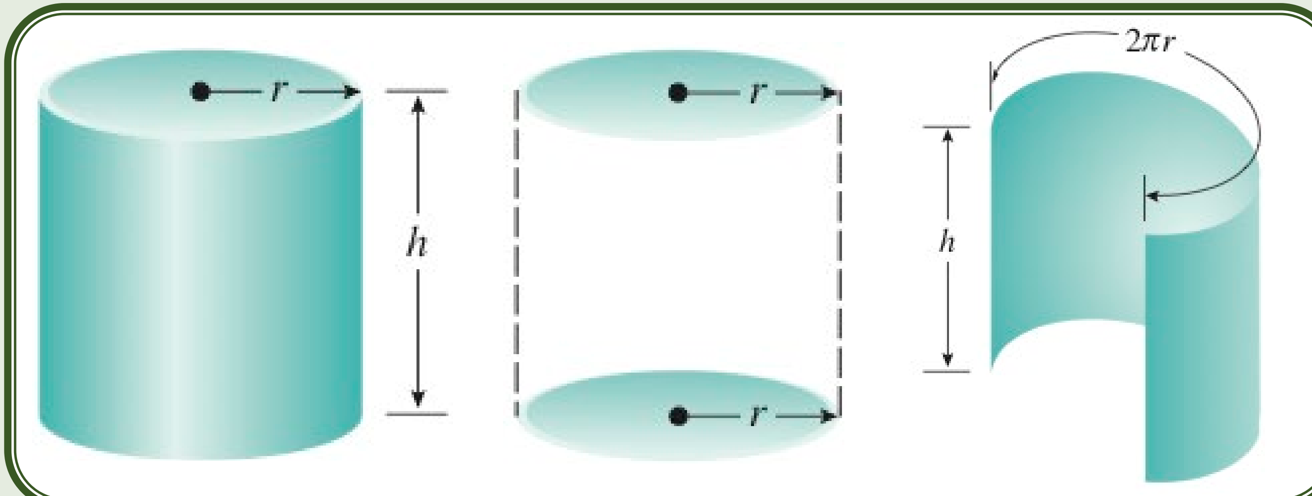
The inscribed cylinder of largest volume has radius, r = _____ and height, h = ____.

Ans: $0 \leq r \leq 6$

Maximum and Minimum

Exercise

A closed cylindrical can is to hold 1 liter (1000 cm^3) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can? ***Hint:*** *no absolute bound (use second derivative)*



$$Ans: h = 2r;\ r = \frac{10}{\sqrt[3]{2\pi}} \approx 5.4$$

Curve Sketching

Curve Sketching of Polynomial Function

Curve Sketching of Polynomial Function

These are the several steps to be followed in sketching the graph of polynomial function $y = f(x)$.

Step 1	Find any obvious point when $x = 0$ or $y = 0$.
Step 2	Find $f'(x)$ and $f''(x)$.
Step 3	Put $f'(x) = 0$, then solve the equation to find the critical numbers of x. Find $f(x)$ for each critical number and plot these critical points.
Step 4	Use the Second Derivative Test to find any local maximum and minimum points. For maximum: $f'(x) = 0, f''(x) < 0$. For minimum: $f'(x) = 0, f''(x) > 0$. If $f''(x) = 0$, use the First Derivative Test to determine the local maximum and minimum point.

Curve Sketching of Polynomial Function

Step 5	*Determine the values of x for $f''(x) = 0$. Use these values of x to divide x-axis into intervals. Test the concavity of the point in the interval.* *If $f''(x) > 0$, graph is concave.* *If $f''(x) < 0$, graph is convex.*
Step 6	*Use the information in **Step 5** to find any points of inflection and plot the points.*
Step 7	*Determine any horizontal, vertical or oblique asymptotes.*
Step 8	*List all the points in the table and use this information to determine a suitable scale of the graph. Draw a curve at each local maximum and minimum points. Sketch the graph starting from the points of inflection. Plot any additional points if necessary.*

Curve Sketching of Polynomial Function

Example 12

Draw the graph of the function $f(x) = x^3 - 12x + 3$

Step 1	When $x = 0$, $f(0) = 3$
Step 2	$f'(x) = 3x^2 - 12$, $f''(x) = 6x$.
Step 3	The critical numbers are obtained when $f'(x) = 0$, that is $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 0$. The critical numbers are -2 and 2. Therefore the critical points are $(-2, 19)$ and $(2, -13)$.
Step 4	Use the Second Derivative to determine whether the critical points are local maximum or minimum points. $f''(-2) = 6(-2) = -12 < 0$. → We obtain a local maximum at $x = -2$ $f''(2) = 6(2) = 12 > 0$. → We obtain a local minimum at $x = 2$

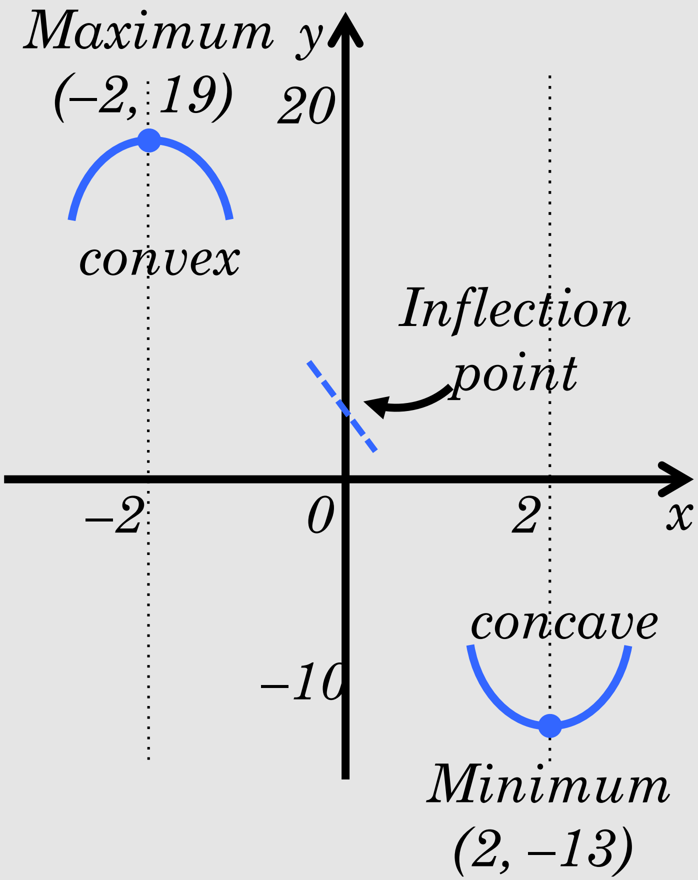
Curve Sketching of Polynomial Function

Example 12

Step 5	*Let $f''(x) = 0$ that is $f''(x) = 6x = 0$.* *At $x = 0$, perhaps it is a critical number.* *Test using a point a where $x < 0$, $f''(-1) < 0$.* *→The graph is convex for $x < 0$.* *Test using a point a where $x > 0$, $f''(1) > 0$.* *→The graph is concave for $x > 0$.*
Step 6	*At $x = 0$, the graph has a point of inflection because $f(0)$ exists, $f''(0) = 0$ and concavity changes* *$f(0) = (0)^3 - 12(0) + 3 = 3$* *Write all the information in the following graph*

Curve Sketching of Polynomial Function

Step 7	*Since f(x) is a polynomial so there are no asymptotes.*
Step 8	*Draw* *x–axis: x = –4 to x = 4* *y–axis: x = –13 to x = 19* *Plot several additional points to determine the shape of the graph*

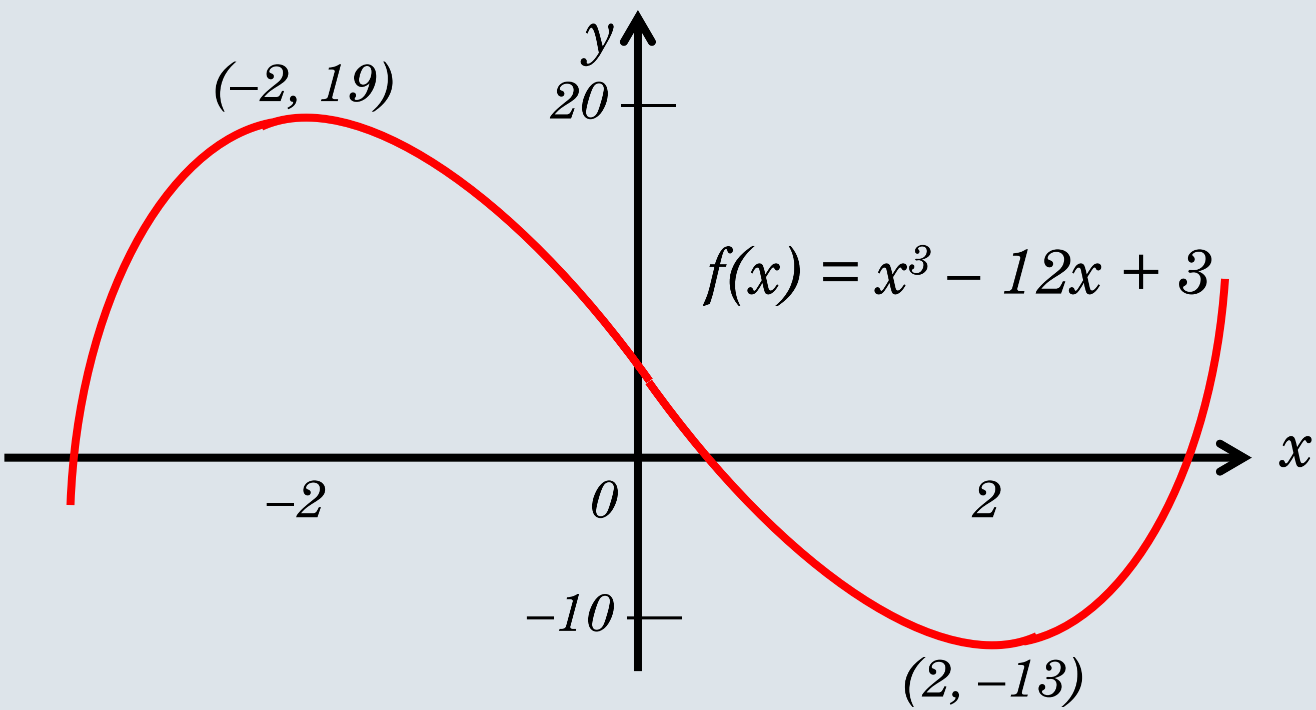


Example 12

Curve Sketching of Polynomial Function

Example 12

Values of x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-13	12	19	14	3	-8	-13	-6	19



Graph Sketching of Rational Functions

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Graph Sketching of Rational Functions

Definition

Vertical and Horizontal Asymptotes

If f(x) approaches positive or negative infinity when x is approaching x_0, and if

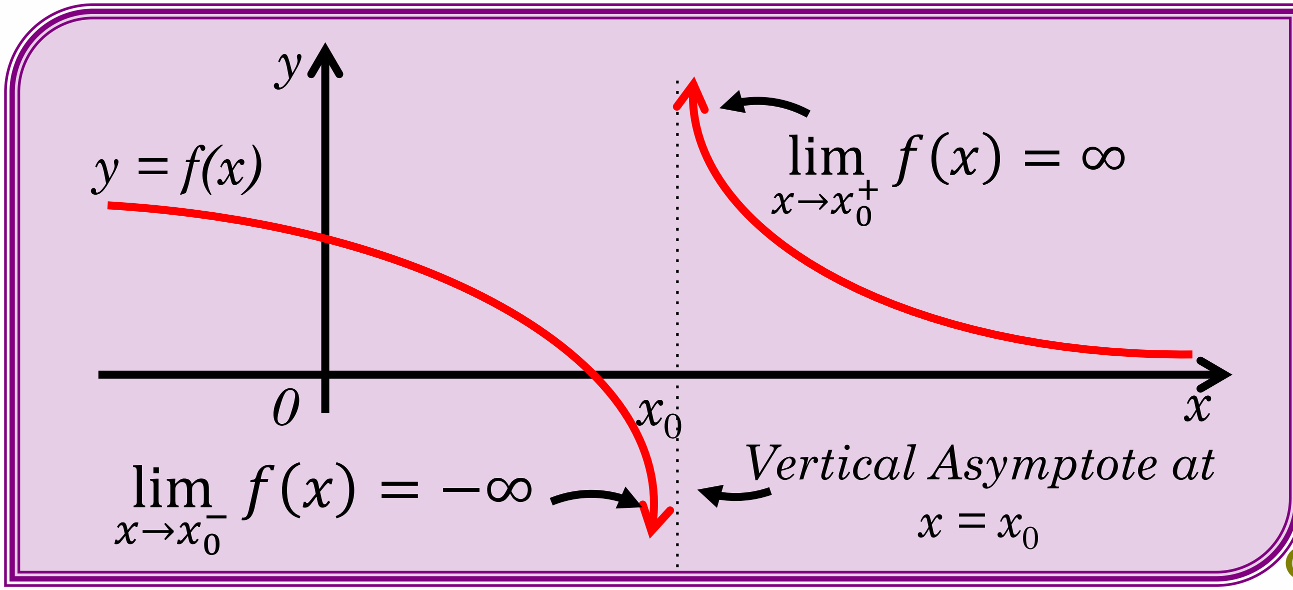
$$\lim_{x \to x_0^-} f(x) = \pm\infty \quad or \quad \lim_{x \to x_0^+} f(x) = \pm\infty$$

*Then the line $x = x_0$ is called a **vertical asymptote** of $f(x)$.*

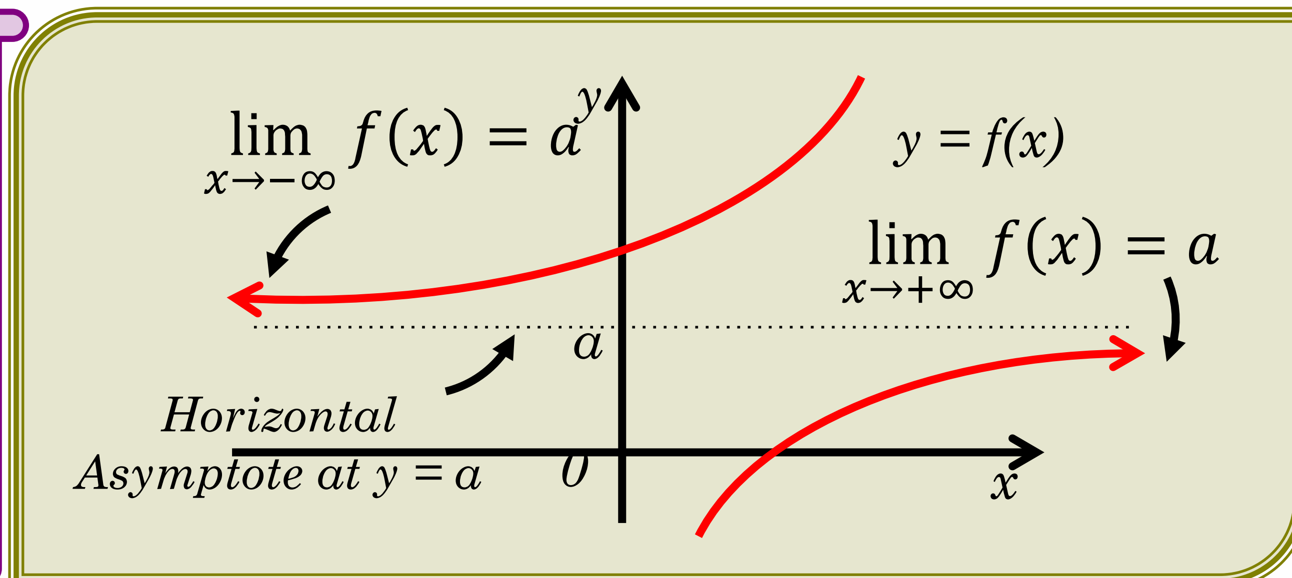
*If f(x) approaches values of a when x is $x \to +\infty$ or $x \to -\infty$, then the line $y = a$ is called a **horizontal asymptote** of $f(x)$.*

Graph Sketching of Rational Functions

Vertical Asymptote



Horizontal Asymptote



Graph Sketching of Rational Functions

Example 13

Find the vertical and horizontal asymptotes of the function.

$$f(x) = \frac{x-1}{x-2}$$

Vertical solution:

Equate the denominator to zero.

$$x - 2 = 0 \Rightarrow x = 2$$

Then f(x) is not continuous at x = 2. To find the vertical asymptote, need to check $\lim_{x\to 2} f(x)$

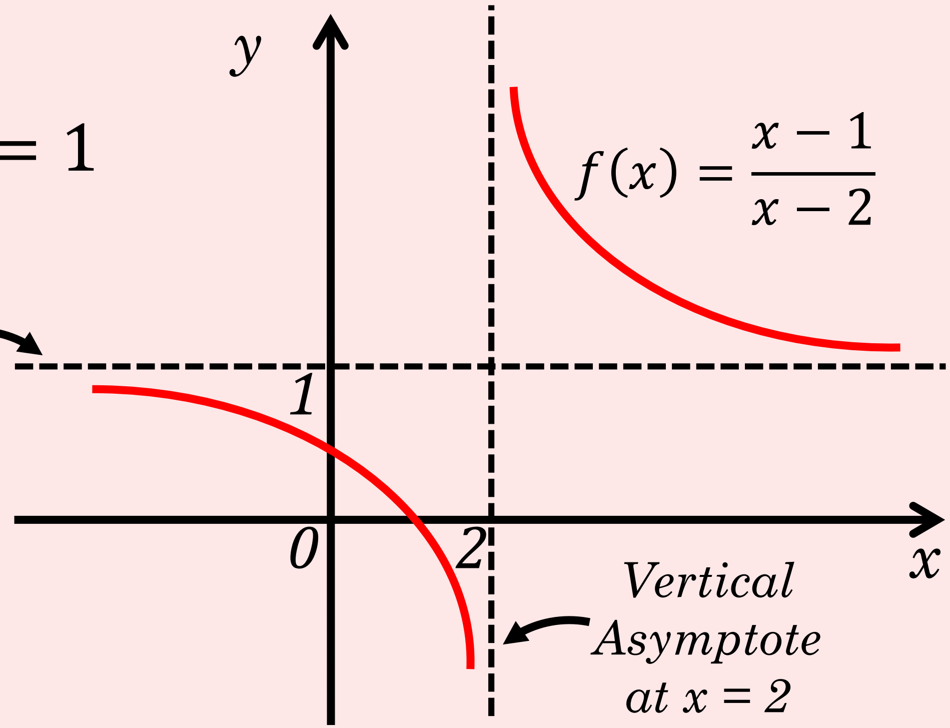
$$\lim_{x\to 2^-} \frac{x-1}{x-2} = -\infty \quad and \quad \lim_{x\to 2^+} \frac{x-1}{x-2} = +\infty$$

Graph Sketching of Rational Functions

Horizontal solution:

To find the horizontal asymptote, need to check $\lim_{x\to\infty} f(x)$. *Therefore*

$$\lim_{x\to\infty}\frac{x-1}{x-2}=\lim_{x\to\infty}\frac{1-{}^{1}/_{x}}{1-{}^{2}/_{x}}=1$$



Example 13

Graph Sketching of Rational Functions

Definition

Oblique Asymptote

A straight line $y = mx + c$, where m and c are constant, is an **oblique asymptote** of $f(x)$ if and only if

$$\lim_{x \to +\infty} f(x) = mx + c$$

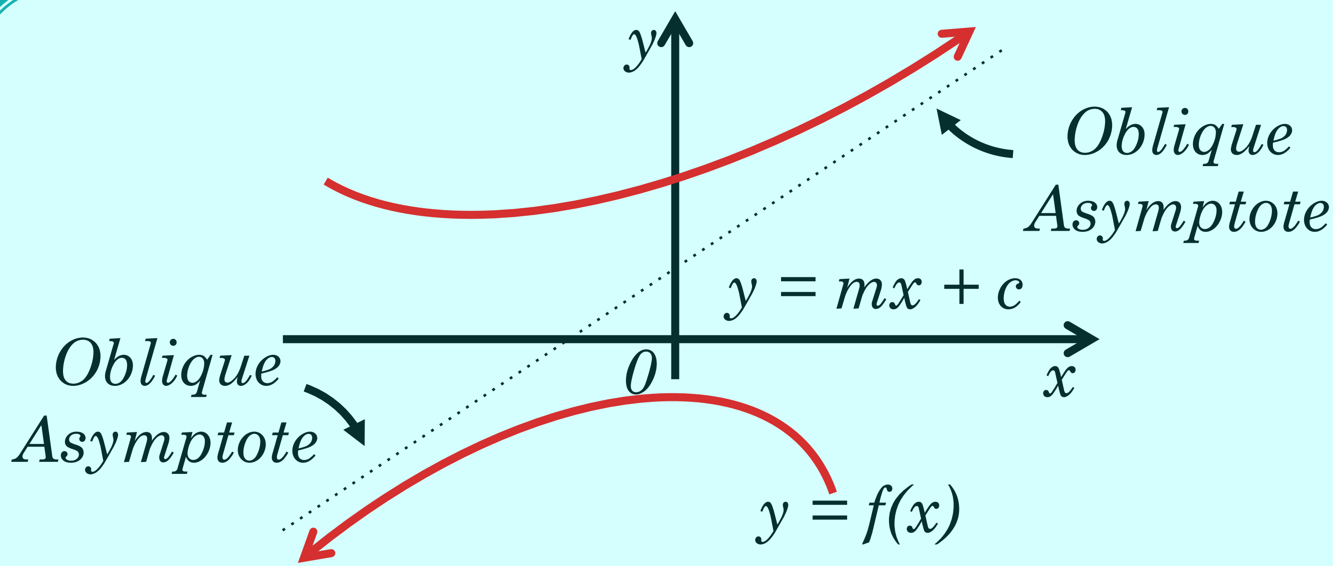
The values m and c can be obtained from

$$m = \lim_{x \to +\infty} \frac{f(x)}{x}$$

$$c = \lim_{x \to +\infty} \{f(x) - mx\}$$

Also true for $x \to -\infty$.

Oblique Asymptote



Maximum and Minimum

Final Sem 1 20182019

a) The derivative of a function $y = f(x)$ is given as

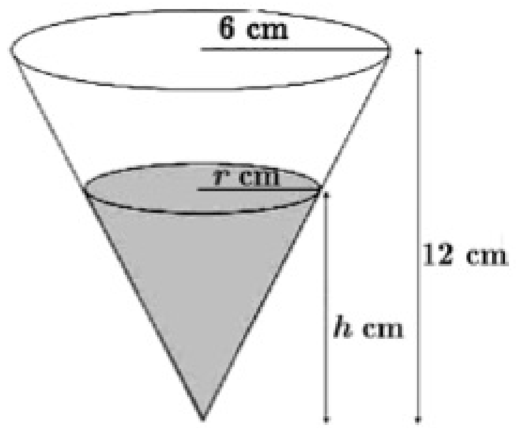
$$\frac{dy}{dx} = \frac{x^2 - 4x - 5}{x - 2}.$$

i. Find all the critical values of the function.

(4 marks)

ii. Use First Derivative Test to determine which critical values will give maximum or minimum points.

(4 marks)



Maximum and Minimum

Final Sem 1 20182019

b) An empty cone of radius 6 cm and height 12 cm is being filled with water. At certain instant, the radius of water is r cm and height h cm.

i. Show that the volume of water is

$$V = \frac{\pi}{12}h^3.$$

(3 marks)

ii. If the volume flow rate of water is $20\,\text{cm}^3 s^{-1}$, find the rate of water level is rising when $h = 2$ cm.

(4 marks)