CHAPTER 4
INTEGRATION

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4.6.3 Even Powers of sin x and cos x

Anti-derivatives and Indefinite Integrals

Anti-derivatives and Indefinite Integrals

A function $F(x)$ is an anti-derivative of $f(x)$ if

$$F'(x) = f(x) \text{ for all } x \in D_f$$

For example if $F(x) = x^4$, then $f(x) = 4x^3$ so that x^4 is an anti-derivative of $4x^3$. Since $\frac{d}{dx}[F(x) + c] = f(x)$, c is a constant, $F(x) + c$ is an anti-derivative of $f(x)$.

Anti-derivatives and Indefinite Integrals

A set of all anti-derivatives of $f(x)$ is called indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

- Integral symbol: $\int$
- Integrand of the integral: $f(x)$
- Variable of integration: x

If $F'(x) = f(x)$ then

$$\int f(x)\,dx = F(x) + c$$

Anti-derivatives and Indefinite Integrals

or $\int F'(x)\,dx = F(x) + c$, *where c is constant of integration.*

The relationship between derivative and indefinite integral formulae for some elementary functions

Anti-derivatives and Indefinite Integrals

	Derivative Formula	Equivalent Integral Formula
1.	$\frac{d}{dx}(x^3) = 3x^2$	$\int 3x^2\,dx = x^3 + c$
2.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
3.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
4.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + c$
5.	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln x + c$

Anti-derivatives and Indefinite Integrals

Determine whether the following results are correct:

(i) $$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx = -\frac{\sqrt{x^2+1}}{x^2} + c$$

(ii) $$\int xe^x\,dx = xe^x - e^x + c$$

(iii) $$\int \frac{1}{\sec x}\,dx = \sin x + c$$

(iv) $$\int \frac{\sin x}{\cos^2 x}\,dx = \sec x + c$$

Example 1

Ans: F, T, T, T

Anti-derivatives and Indefinite Integrals

Example 2

1) *Differentiate* $(x-1)e^x$ *with respect to x. Hence evaluate* $\int xe^x\,dx$.

2) *If* $e^x \sin x = \dfrac{d}{dx}\{e^x(A\sin x + B\cos x)\}$ *, find the value of A and B. Hence evaluate* $\int e^x \sin x\,dx$.

3) *Show that* $\dfrac{d}{dx}(\tan^3 x) = 3\sec^4 x - 3\sec^2 x$.

Hence evaluate $\int \sec^4 x\,dx$

Standard Integrals

Standard Integrals

(i)	$\int dx = x + c$
(ii)	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
(iii)	$\int e^x \, dx = e^x + c$
(iv)	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$
(v)	$\int \sin x \, dx = -\cos x + c$

Standard Integrals

(vi)	$\int \cos x \, dx = \sin x + c$
(vii)	$\int \sec^2 x \, dx = \tan x + c$
(viii)	$\int \operatorname{cosec}^2 x \, dx = -\operatorname{cotan} x + c$
(ix)	$\int \sec x \tan x \, dx = \sec x + c$
(x)	$\int \operatorname{cosec} x \operatorname{cotan} x \, dx = -\operatorname{cosec} x + c$

Standard Integrals

Example 3

Evaluate the following integrals

(i) $\int x^{10}\, dx$

(ii) $\int \frac{1}{x^8}\, dx$

(iii) $\int x^{-\frac{2}{3}}\, dx$

(iv) $\int \frac{\sin 2x}{2\sin x}\, dx$

Basic Properties of Indefinite Integrals

Basic Properties of Indefinite Integrals

(i)	$\int kf(x)\,dx = k\int f(x)\,dx,$ k is a constant
(ii)	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Basic Properties of Indefinite Integrals

Example 4

Evaluate the following integrals

i) $\displaystyle\int x^3 + \frac{2}{x^3}\,dx$

ii) $\displaystyle\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx$

iii) $\displaystyle\int (x + 1)(3x - 2)\,dx$

iv) $\displaystyle\int (e^x - 4\tan^2 x)\,dx$

v) $\displaystyle\int \sin^2 \frac{x}{2}\,dx$

Definite Integrals

Definite Integrals

Notation

$$\int_a^b f(x)\,dx$$

Upper Limit: b

Lower Limit: a

Integrand of the Integral: $f(x)$

Definite Integrals

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where

$$F'(x) = f(x) \text{ and } x \in [a, b]$$

Definite Integrals

If $f(x)$ and $g(x)$ are continuous functions on the interval $[a, b]$, then

i) $\int_a^a f(x)\,dx = 0$ if $F(a)$ exists.

ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

iii) $\int_a^b kf(x)\,dx = \int_a^b f(x)\,dx$, k is a constant.

iv) $\int_a^b k\,dx = k(b-a)$.

Basic Properties of Definite Integrals

Definite Integrals

If f(x) and g(x) are continuous functions on the interval [a, b], then

v) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$

where $a \le c \le b$

vi) $\int_a^b [f(x) \pm g(x)]\,dx =$

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

Basic Properties of Definite Integrals

Definite Integrals

Evaluate the following integrals

Example 5

i) $\int_{4}^{9} x^{-\frac{1}{2}}\, dx$

ii) $\int_{16}^{25} \left(\frac{2}{\sqrt{x}} - 3\sqrt{x}\right) dx$

iii) $\int_{1}^{2} \frac{x^3 - 2x^2 + 4}{x^3}\, dx$

iv) $\int_{1}^{2} (x+2)(x+3)\, dx$

v) $\int_{-2}^{3} |4x - 6|\, dx$

vi) $\int_{-1}^{1} (|2x| - x)\, dx$

Definite Integrals

Example 6

a) If $y = x\sqrt{x+1}$, show that $\dfrac{dy}{dx} = \dfrac{3x+2}{2\sqrt{x+1}}$

Hence evaluate $\displaystyle\int_3^8 \frac{3x+2}{\sqrt{x+1}}\,dx.$

b) Differentiate $y = x\sqrt{3x^2+6}$ with respect to x.

Hence evaluate $\displaystyle\int_1^5 \frac{x^2+1}{2\sqrt{3x^2+6}}\,dx.$

Definite Integrals

c) *Show that* $\frac{d}{dx}(\tan^3 x) = 3\tan^4 x + 3\sec^2 x - 3$

Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx.$

Example 6

Techniques of Integration

Techniques of Integration

Integration by Substitution

$$\int f(g(x))g'(x)\,dx$$

Step 1 Let $u = g(x)$

Step 2 Obtain $\dfrac{du}{dx} = g'(x)$

Step 3 Substitute $u = g(x)$ and $du = g'(x)$

The whole integral must be in term of u

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Integration by Substitution

$$\int f\big(g(x)\big)g'(x)\,dx$$

Step 4 Evaluate the integral must be in term of u.

Step 5 Substitute $u = g(x)$, so the final must be in terms of x.

Integration by Substitution

Evaluate the following integrals

i) $\int (2x+1)^{10}\,dx$

ii) $\int \frac{1}{\sqrt{2x+1}}\,dx$

iii) $\int \frac{x^3}{(1+x^4)^{\frac{1}{3}}}\,dx$

iv) $\int xe^{3x^2}\,dx$

v) $\int e^x\sqrt{1+e^x}\,dx$

vi) $\int \frac{e^x}{e^x+4}\,dx$

Example 7

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Integration by Substitution

Use substitution method to evaluate the following integrals.

i) $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx$

ii) $\displaystyle\int \frac{1}{x\ln x^2}\,dx$

iii) $\displaystyle\int \frac{\ln x}{x\sqrt{1+\ln x}}\,dx$

iv) $\displaystyle\int \tan 2x\,dx$

v) $\displaystyle\int \sin\left(2x+\frac{\pi}{3}\right)dx$

vi) $\displaystyle\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$

Example 8

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Integration by Substitution

Example 9

Use substitution method to evaluate the following integrals.

i) $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx$

ii) $\displaystyle\int x\cos(x^2)\,e^{\sin(x^2)}\,dx$

iii) $\displaystyle\int (x+1)\sin(x^2+2x)\,dx$

iv) $\displaystyle\int x^2 e^{x^3+2}\,dx$

v) $\displaystyle\int \sin x\cos^5 x\,dx$

vi) $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$

Integration by Substitution

Evaluate the following integrals using the given substitutions.

i) $\int x(x-2)^5\,dx \; ; \; u = x - 2$

ii) $\int x\sqrt{x+3}\,dx \; ; \; u = x + 3$

iii) $\int \frac{x-1}{(2x-1)^2}\,dx \; ; \; u = 2x - 1$

iv) $\int \frac{x}{(2x+1)}\,dx \; ; \; u = 2x + 1$

Example 10

Techniques of Integration

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate with respect to x

$$\frac{d}{dx}(uv) = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

or

$$uv = \int u\,dv + \int v\,du$$

So that

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts

Integration by Parts

Example 11

Evaluate the following integrals using by parts.

i) $\int xe^x \, dx$

ii) $\int x \sin x \, dx$

iii) $\int \ln x \, dx$

iv) $\int x \ln x \, dx$

v) $\int x \cos x \, dx$

vi) $\int e^x \sin x \, dx$

Integration by Parts

Use integration by parts to evaluate the following integrals.

i) $\int x^3 \ln 5x \, dx$

ii) $\int x^5 e^{x^3} \, dx$

iii) $\int \frac{x^3 e^{x^2}}{(x^2+1)^2} \, dx$

iv) $\int \frac{x^3}{(x^2+5)^2} \, dx$

Example 12

Techniques of Integration

Tabular Method

- *Special case of integration by parts*
- *Integrals should be in the form*

$$\int x^n e^{ax}\,dx \qquad \int x^n \sin ax\,dx$$

$$\int x^n \cos ax\,dx$$

- *To solve* $\int uv'\,dx$ *where* $dv = v'\,dx$*, the integrals must satisfy the following conditions:*

a) u can be differentiated repeatedly with respect to x until becoming 0.

b) v' can be integrated repeatedly with respect to x easily.

Tabular Method

Example 13

Evaluate the following integrals using the tabular method.

i) $\int x^4 e^{2x}\,dx$

ii) $\int x^2 \cos 3x\,dx$

iii) $\int x^3 \sin 2x\,dx$

iv) $\int x^2 e^{-3x}\,dx$

Techniques of Integration

Integral should be in the form

$$\int \frac{M(x)}{N(x)}\,dx$$

where M(x) and N(x) are polynomials. Degree of M(x) must be less than degree of N(x). Otherwise, we need to perform long division.

Integration by Partial Fraction

Example 14

Evaluate the following integrals.

i) $\int \frac{x}{x-1}\,dx$

ii) $\int \frac{x^2}{x+1}\,dx$

iii) $\int \frac{x^2}{x^2+3x+2}\,dx$

iv) $\int \frac{x^3}{x^2+5x+6}\,dx$

Integration by Partial Fraction

Evaluate the following integrals.

Example 15

i) $\displaystyle\int \frac{1}{(x-1)(x-2)}\,dx$

iii) $\displaystyle\int \frac{2-3x}{(3x^2-4x+1)}\,dx$

ii) $\displaystyle\int \frac{x}{(x+1)(x+2)^2}\,dx$

iv) $\displaystyle\int \frac{x+1}{(1-x)(x^2+1)}\,dx$

Integration by Partial Fraction

Evaluate the following integrals using the given substitution.

i) $\displaystyle\int \frac{1}{1-e^x}\,dx\,;\quad u = e^x$

ii) $\displaystyle\int \frac{1}{e^x(1-e^x)}\,dx\,;\quad u = 1-e^x$

iii) $\displaystyle\int \frac{1}{e^x(1+e^x)}\,dx\,;\quad u = 1+e^x$

Example 16

Techniques of Integration

Odd powers of sin x and cos x

Use identity

$$\sin^2 x + \cos^2 x = 1$$

Integration of Trigonometric Functions

Integration of Trigonometric Functions

Example 17

Evaluate the following integrals.

i) $\int \sin^3 x \, dx$

iii) $\int \sin^3 x \cos^2 x \, dx$

ii) $\int \cos^5 x \, dx$

iv) $\int \cos^5 x \sin^2 x \, dx$

Techniques of Integration

Products of sin ax cos bx, sin ax sin bx and cos ax cos bx.

Use identity

- $\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$
- $\sin ax \sin bx = -\frac{1}{2}[\cos(a+b)x - \cos(a-b)x]$
- $\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$

Integration of Trigonometric Functions

Integration of Trigonometric Functions

Example 18

Evaluate the following integrals.

i) $\int \sin 4x \cos 2x \, dx$

ii) $\int \sin 2x \sin 7x \, dx$

iii) $\int \cos 3x \cos 8x \, dx$

Techniques of Integration

Even powers of $\sin x$ and $\cos x$

Use identity

- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

Integration of Trigonometric Functions

Example 19

Evaluate the following integrals.

i) $\int \sin^2 x \, dx$

ii) $\int \cos^4 x \, dx$

Exercises

1. Determine whether the following results are correct.

i) $$\int x^3\sqrt{2-x^2}\,dx = \left(\frac{2-x^2}{5}-\frac{2}{3}\right)\sqrt{(2-x^2)^3}+c$$

ii) $$\int \left(\frac{1}{x}+\frac{2x}{2x^2+1}\right)dx = \ln\left|x\sqrt{2x^2+1}\right|+c$$

2. Find each of the following indefinite integral.

a) $$\int \frac{2x^2-4}{\sqrt{x}}\,dx$$

b) $$\int \left(\frac{3}{(2x+1)^3}+\sqrt{1-2x}\right)dx$$

c) $$\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx$$

d) $$\int x\sqrt{5+x}\,dx$$

Exercises

e) $\int \frac{x^4}{x^5+1}\,dx$

f) $\int \frac{\cos x}{\sqrt{\sin x}}\,dx$

g) $\int \frac{\cos 3x}{1+\sin 3x}\,dx$

h) $\int (\sec^2 5x)(1+5x)\,dx$

3. *Use integration by parts to evaluate the following integrals.*

a) $\int xe^{-x}\,dx$

b) $\int x\cos 2x\,dx$

c) $\int x\sec^2 x\,dx$

d) $\int (\ln x)^2\,dx$

e) $\int x^2\ln x\,dx$

f) $\int e^x\cos x\,dx$

Exercises

4. Use partial fraction to evaluate the following integrals.

a) $\int \frac{2x+3}{x^2-9}\,dx$

b) $\int \frac{1}{x^2-4}\,dx$

c) $\int \frac{x^2-1}{x^2-16}\,dx$

d) $\int \frac{x^2+x-1}{x(x^2-1)}\,dx$

e) $\int \frac{x^2-x+1}{(x+1)^3}\,dx$

f) $\int \frac{2x+4}{x^3-2x^2}\,dx$

5. Compute the given definite integrals.

a) $\int_0^1 \left(2\sqrt{x}+x^2-2\right)dx$

b) $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$

c) $\int_1^2 \frac{(e^x+e^{-x})^2}{e^x}\,dx$

d) $\int_0^1 \frac{x+1}{(x+2)(x+3)}\,dx$

Exercises

e) $\displaystyle\int_1^2 \frac{x^4 + 2x^2 + 1}{x^2}\,dx$

g) $\displaystyle\int_5^8 \frac{1}{\sqrt{3x+1}}\,dx$

f) $\displaystyle\int_{-3}^4 |x+1|\,dx$

h) $\displaystyle\int_0^1 x\ln(1+x^2)\,dx$

6. Evaluate.

a) $\displaystyle\int_1^2 \frac{x+1}{x(2x+1)}\,dx$

c) $\displaystyle\int_1^2 \frac{x(2x+1)}{(x+1)}\,dx$

b) $\displaystyle\int_{-2}^1 \frac{1}{x - x\ln x}\,dx$

d) $\displaystyle\int_0^\pi \frac{x}{1+x^2}\,dx$

Exercises

7. Given $\frac{x^2-2x-9}{(2x-1)(x^2+3)} \equiv \frac{A}{(2x-1)} + \frac{Bx+C}{(x^2+3)}$, show that $C = 0$ and obtain the values of A and B. Hence evaluate

$$\int_1^2 \frac{x^2-2x-9}{(2x-1)(x^2+3)}\,dx$$

giving your answer correct to two significant figures.

8. Prove that $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$.

   Then evaluate $\int_4^9 \frac{1}{x^2-4}\,dx$.

9. By substituting $u^2 = x+1$, determine $\int \frac{1}{x\sqrt{x+1}}\,dx$.

Exercises

10. Express $\dfrac{5x^3 - x^2 + x - 1}{x^2(x^2+1)}$ in the form of partial fraction.

Hence evaluate

$$\int_1^e \frac{5x^3 - x^2 + x - 1}{x^2(x^2+1)}\,dx$$

11. By using a suitable substitution, find

$$\int \frac{\cos 3x}{\sin^2 3x}\,dx$$

12. Evaluate each of the following integrals.

a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 5x \cos x \, dx$

b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 x \cos^2 x \, dx$

Exercises

c) $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \cot x \, dx$

d) $\int \frac{\cos^3 \theta}{\sin^2 \theta} d\theta$

13. *Show that* $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$, *Hence evaluate*

$$\int_0^{\frac{\pi}{4}} \left(\frac{1}{1-\sin 3x} + \frac{1}{1+\sin 3x} \right) dx$$

14. *Find* $\int \sqrt{\sin x} \cos^3 x \, dx$, *Hence evaluate* $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos^3 x \, dx$

15. *By using partial fraction, show that*

$$\int \frac{x^2+1}{x^2-4} dx = Ax + B \ln\left|\frac{x-2}{x+2}\right| + C$$

Exercises

Hence

a) *Find a and b*

b) *Evaluate* $\displaystyle\int_3^{10} \frac{x^2+1}{x^2-4}\,dx$