CHAPTER 1

INTRODUCTION AND MATHEMATICAL PRELIMINARIES

1.1 FLUID MECHANICS

Mechanics is the oldest physical science that deals with both the stationary and moving bodies under the influence of forces. The branch of mechanics that deals with the bodies at rest is called **statics**, while the branch deals with bodies in motion is called **dynamics**. **Fluid mechanics** is defined as the science that deals with the behaviour of fluids at rest or in motion and the interaction of fluids with solids or other fluids at the boundaries.

Fluid mechanics itself is also divided into several categories.

a) **Hydrodynamics** – the study of the motion of fluids that are practically incompressible such as liquids especially water and gases at low speeds

   Hydraulics – a subcategory of hydrodynamics which deals with liquid flows in pipes and open channels.

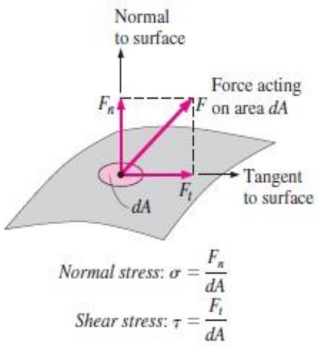
b) **Gas dynamics** – it deals with the flow of fluids that undergo significant density changes, such as the flow of gases through nozzles at high speeds

c) **Aerodynamics** – deals with the flow of gases especially air over bodies such as aircraft, rockets and automobiles at high or low speeds.

Some other specialized categories such as **meteorology, oceanography,** and **hydrology** deal with naturally occurring flows.

1.1.1 What is fluid?

We have already learnt that a substance exists in three primary phases: solid, liquid, and gas. At very high temperatures, it also exists as plasma. A substance in the liquid or gas phase is referred to as a fluid. Distinction between a solid and a fluid is made on the basis of the substance's ability to resist an applied shear (or tangential) stress that tends to change its shape. A solid can resist an applied shear stress by deforming, whereas a fluid deforms continuously under the influence of shear stress, no matter how small.



Stress is defined as force per unit area and is determined by dividing the force by the area upon which it acts. The normal component of the force acting on a surface per unit area is called the **normal stress**, and the tangential component of a force acting on a surface per unit area is called **shear stress**. In a fluid at rest, the normal stress is called **pressure**. The normal stress and shear stress at the surface of a fluid element. For fluids at rest, the shear stress is zero and pressure is the only normal stress.

1.1.2 Classification of fluids

The fluid may be classified into the following five types:

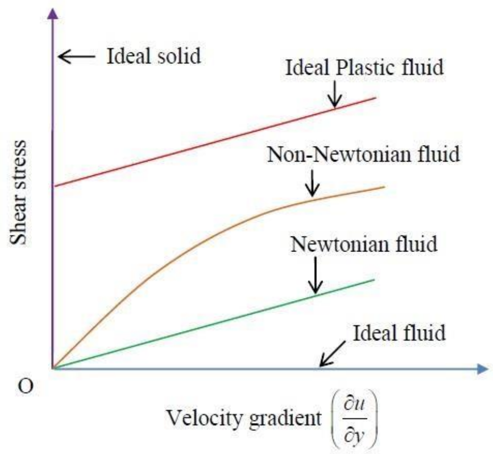


Fig 1.1 Stress strain graph of different types of fluid

a) Ideal Fluid

A fluid, which is incompressible and is having no viscosity, is known as ideal fluid. Ideal fluid is only an imaginary fluid because all the fluids, which exit, have some viscosity.

b) Real Fluid

A fluid which possesses viscosity is known as real fluid. All the fluids in practice are real fluids.

c) Newtonian fluid

A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as the Newtonian fluid. Example:

d) Non-Newtonian Fluid

A real fluid, in which the shear stress is not proportional to the rate of shear strain (or velocity gradient), is known as the non-Newtonian fluid. Example:

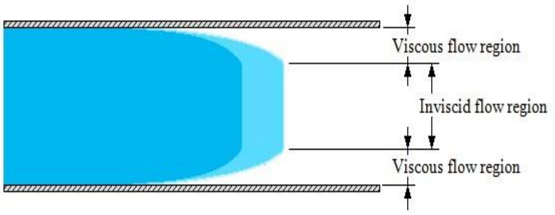
e) Ideal-Plastic Fluid

A fluid, in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient), is known as ideal plastic fluid. Example:

1.1.3 Types of fluid flow

There are many ways to classify fluid flow problems, and here we discuss some general categories.

a) Viscous and Inviscid Regions of Flow



When two fluid layers move relatively to each other, a frictional force develops between them which is quantified by the fluid property *viscosity*. The fluid flow in which the frictional effects become signification are treated as **viscous flows.** An **inviscid flow** is a flow where viscosity is not important, there is no shear force between adjacent fluid layers.

b) Internal and External Flow

A fluid flow is classified as being internal or external, depending on whether the fluid is forced to flow in a confined channel or over a surface. The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is **external flow.** The flow in a pipe or duct is **internal flow** if the fluid is completely bounded by solid surfaces.

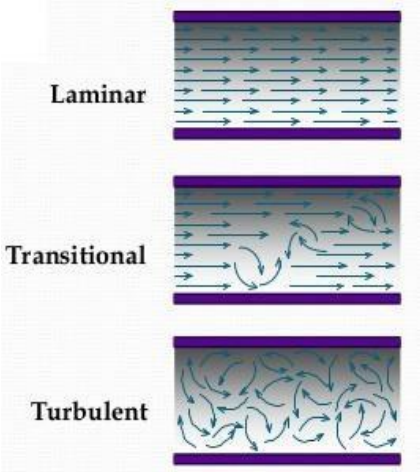
Eg: Internal flow -

External flow -

c) Compressible and Incompressible Flow

A flow is classified as being *compressible* or *incompressible*, depending on the level of variation of density during flow. Incompressibility is an approximation, and a flow is said to be **incompressible** if the density remains nearly constant throughout. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion when the flow is incompressible. The densities of liquids are essentially constant, and thus the flow of liquids is typically *incompressible*. When the density variation during a flow is more than 5% then it is treated as **compressible**. Gases are highly *compressible*.

d) Laminar and Turbulent Flow



The highly ordered fluid motion characterized by smooth layers of fluid is called **laminar**. The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called **turbulent**. The flow of low-viscosity fluids such as air at high velocities is typically turbulent. A flow that alternates between being laminar and turbulent is called **transitional.**

e) Steady and Unsteady flow:

When there is no change in fluid property at point with time, then it implies as **steady flow**. However, the fluid property at a point can also vary with time which means the flow is **unsteady/transient**. The term ‘periodic' refers to the kind of unsteady flows in which the flow oscillates about a steady mean.

f) Natural and Forced flow:

In a forced flow, the fluid is forced to flow over a surface by external means such as a pump or a fan. In natural flow, density difference is the driving factor of the fluid flow. Here, the buoyancy plays an important role. For example, a warmer fluid rises in a container due to density difference.

g) One/Two/Three-dimensional flow:

A flow field is best characterized by the velocity distribution, and thus can be treated as one/two/three-dimensional flow if velocity varies in the respective directions.

1.2 BACKGROUND KNOWLEDGE

a) Vectors

Fluid dynamics is about the motion of fluids, so velocities must come in; that is the whole subject will be full of work with vectors. You must be confident in your use of the two products and in the use of components and unit vectors

Dot product	$\vec{A}\cdot\vec{B}$
Cross product	$\vec{A}\times\vec{B}$

These components may be those appropriate to cartesian axes or polar directions.

b) Function of several variables

The velocities in fluid dynamics will in general depend on position and time, e.g

$$\hat{u}=u(x,y,z,t)\hat{i}+v(x,y,z,t)\hat{j}+w(x,y,z,t)\hat{k}$$

c) Vector calculus

Since the velocities change in space and time, we shall need vector calculus. For example,

the divergence of $\hat{v}$, $\nabla\cdot\hat{v}$ or div $\hat{v}$

and the curl of $\hat{v}$, $\nabla\times\hat{v}$ or curl $\hat{v}$

d) Solution of ODE and PDE

- Characteristics equation
- Euler solution
- Separation of variable method
- Principle of superposition
- Linear and nonlinear equation

1.3 POLAR COORDINATE SYSTEMS

a) Plane polar coordinate

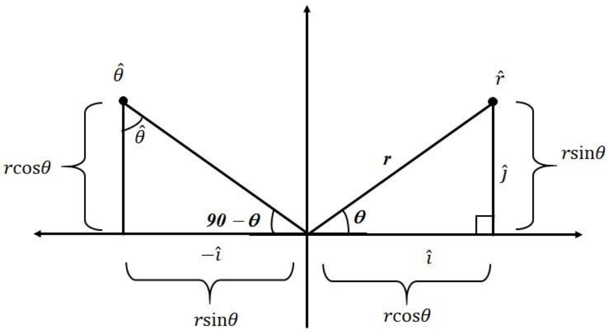


Fig 1.2 Plane polar coordinate

Plane polar coordinate (r, θ) are related to Cartesian coordinates (x, y) by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and the unit vectors $\hat{i}, \hat{j}$ of Cartesians are related to those for polars, $\hat{r}$ and $\hat{\theta}$, by

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$
$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

If you calculate $d\hat{r}/d\theta$ and $d\hat{\theta}/d\theta$ from these formulae, you get

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$
$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

Clearly also

$$\frac{d\hat{r}}{dr} = \frac{d\hat{\theta}}{dr} = 0$$

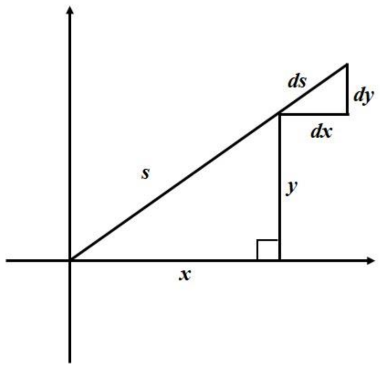


Fig 1.3 Length elements in cartesian and plane polar coordinates

The length element ds is given by

$$ds^2 = dx^2 + dy^2$$

We know that,

$$x = r\cos\theta, \quad y = r\sin\theta,$$

using total differential,

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$
$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

then, ds^2 in terms of dr^2 and $d\theta^2$ are

$$dx = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$ds^2 = dx^2 + dy^2$$

$=$

$=$

$$ds^2 = dr^2 + r^2 d\theta^2$$

b) Cylindrical coordinate

Cylindrical polar coordinates will be denoted by (r,θ,z). The relations between cartesian and cylindrical polar coordinate are almost exactly as in *a)* above:

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$
$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$
$$\hat{k} = \hat{k}$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

c) Spherical coordinates

Spherical polar coordinates will be donated by (r,θ,λ). The relations to cartesian coordinates and unit vectors are

$$x = r\sin\theta\cos\lambda$$
$$y = r\sin\theta\sin\lambda$$
$$z = r\cos\theta$$

$$\hat{r} = \hat{i}\sin\theta\cos\lambda + \hat{j}\sin\theta\sin\lambda + \hat{k}\cos\theta$$
$$\hat{\theta} = \hat{i}\cos\theta\cos\lambda + \hat{j}\cos\theta\sin\lambda - \hat{k}\sin\theta$$
$$\hat{k} = \hat{i}\sin\lambda + \hat{j}\cos\lambda$$

The length of elements are

$$dr, rd\theta, r\sin\theta d\lambda$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\lambda^2$$

1.4 THE VECTOR DERIVATIVES, ∇

a) Cartesian coordinate

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Let $\phi(x, y, z)$ be a scalar field, then $\nabla\phi$ may be defined as the vector which has cartesian component form

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

b) Plane polar coordinate

$\nabla\emptyset$ has different expression. We may proceed in an elementary fashion by transforming $\partial/\partial x$, $\partial/\partial y$, $\hat{i}, \hat{j}$ to $\hat{r}, \hat{\theta}$ form as follows

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\sin\theta$$
$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\cos\theta$$

by using the chain rule, and

$$\hat{i} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$
$$\hat{j} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

Using these on

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

gives

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

The similar results in cylindrical and spherical polar systems are

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\lambda}\hat{\lambda}$$

1.5 FORMULA INVOLVING ∇

a) $\nabla \cdot \hat{A}$

In cartesian coordinate

If $\hat{A} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\nabla \cdot \hat{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(u\hat{i} + v\hat{j} + w\hat{k} \right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

In plane polar coordinate

If $\hat{A} = U_r\hat{r} + U_\theta\hat{\theta}$

$$\nabla \cdot \hat{A} = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} \right) \cdot \left(U_r\hat{r} + U_\theta\hat{\theta} \right)$$

$=$

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Here,

$\hat{r} \cdot \hat{r} = 1 \quad \hat{\theta} \cdot \hat{\theta} = 1$

$\hat{r} \cdot \hat{\theta} = 0 \quad \hat{\theta} \cdot \hat{r} = 0$

$\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\theta}}{\partial r} = 0$

$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$

In cylindrical coordinate

If $\hat{A} = U_r\hat{r} + U_\theta\hat{\theta} + U_z\hat{k}$

$$\nabla \cdot \hat{A} = \frac{\partial U_r}{\partial r} + \frac{1}{r}U_r + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z}$$

In spherical coordinate

If $\hat{A} = U_r\hat{r} + U_\theta\hat{\theta} + U_\lambda\hat{\lambda}$

$$\nabla \cdot \hat{A} = \frac{\partial U_r}{\partial r} + \frac{1}{r}U_r + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r\sin\theta}U_r\sin\theta + \frac{1}{r\sin\theta}U_\theta\cos\theta + \frac{1}{r\sin\theta}\frac{\partial U_\lambda}{\partial \lambda}$$

b) $\nabla \times \hat{A}$

In cylindrical coordinate

If $\hat{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{k}$

$$\nabla \times \hat{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

In cylindrical coordinate

If $\hat{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\lambda \hat{\lambda}$

$$\nabla \times \hat{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta \hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \lambda} \\ A_r & rA_\theta & r\sin\theta A_\lambda \end{vmatrix}$$

c) $\nabla \cdot \nabla = \nabla^2$

In plane polar coordinate

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} \right) \cdot \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} \right)$$

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In cylindrical coordinate

$$\nabla \cdot \nabla = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In spherical coordinate

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \lambda^2}$$

d) $\nabla^2 \hat{A}$

In plane polar coordinate
If $\hat{A} = A_r \hat{r} + A_\theta \hat{\theta}$

$$\nabla^2 \hat{A} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)\left(A_r \hat{r} + A_\theta \hat{\theta} \right)$$

$=$

In cylindrical coordinate
If $\hat{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{k}$

$$\nabla^2 \hat{A} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right)\left(A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{k} \right)$$

$=$

1.6 CARTESIAN TENSOR METHOD

a) Definition of suffix notation

i. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ where $i = 1, 2, 3$

ii. $\epsilon_{ijk} = 0$ if any two of i, j, k are the same
$\epsilon_{ijk} = +1$ if i, j, k is 123, 231 or 312
$\epsilon_{ijk} = -1$ if i, j, k is 132, 213 or 321

where $i = 1, 2, 3$ $j = 1, 2, 3$ $k = 1, 2, 3$

iii. δ_{ij} (Kronecker delta)

$\delta_{11} = \delta_{22} = \delta_{33} = 1$ if $i = j$
$\delta_{ij} = 0$ if $i \neq j$

iv. $\nabla = \frac{\partial}{\partial x_i}$

v. $\in_{ijk} \in_{klm} = \in_{kij} \in_{klm}$

$$= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

b) Suffix notation for dot and cross product

i. $\vec{a} \cdot \vec{b} = a_i b_i$

ii. $\vec{a} \times \vec{b} = \left(\vec{a} \times \vec{b}\right)_i = \left(\varepsilon_{ijk} a_j b_k\right)_i$

EXERCISE

1. If $\hat{A} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\hat{B} = d\hat{i} + e\hat{j} + f\hat{k}$
   Show that by using suffix notation
   $\hat{A} \times \hat{B} = (bf - ec)\hat{i} + (af - cd)\hat{j} + (ae - bd)\hat{k}$

2. Show that

   i. $\nabla \cdot (\nabla \times \hat{A}) = 0$

   ii. $\nabla \cdot (\phi \hat{A}) = (\nabla \phi \cdot \hat{A}) + \phi \nabla \cdot \hat{A}$

   iii. $\hat{A} \times (\nabla \times \hat{A}) = \nabla \left(\frac{1}{2} A^2\right) - \hat{A} \cdot \nabla \hat{A}$