CHAPTER 2
OBSERVATIONAL PRELIMINARIES

2.1 THE CONTINUOUS MODEL

a) **Fluid as continuum**

The action of forces on fluids can be determine either by:

- Counting the effect of each and every molecule or
- by considering the average effect of the molecules in a given volume, i.e. treating the fluid as a continuum

Continuum can be assumed only when the number of molecules in a small volume (which is much smaller than the body immersed in the fluid) is sufficiently great so that average effects within the volume will be varying smoothly with time.

b) **Material derivatives**

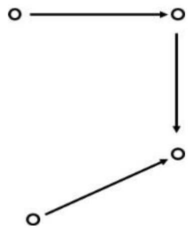
Suppose $\vec{v}(x, y, z, t)$ be a vector function associated with some property of the fluid (eg. Velocity etc.) then

$\frac{D\vec{v}}{Dt}$ or $\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{q} \cdot \nabla \vec{v}$ where $\vec{q} = u, v, w$

$\frac{D\vec{v}}{Dt}$ or $\frac{d\vec{v}}{dt}$ is called the material derivative, $\frac{\partial \vec{v}}{\partial t}$ is called the local derivative and it is associated with time variation at a fixed position

2.2 FLUID VELOCITY AND PARTICLE PATH

Particle path or path lines are the trajectories that individual with fluid particle follows.



If the velocity at some point, $\vec{r}$ of time t is $\vec{v}(\vec{r},t)$, the particle at this point and time move along this velocity vector satisfy

$\dfrac{d\vec{r}}{dt}=\vec{v}(\vec{r},t)$ where $\vec{r}=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$ gives the path of the particle at different time

Example 1:

Given velocity, $\vec{v}=(ay,-ax,0)$, find particles path where a is constant

Solution:

We know that, $\vec{v}=\dfrac{d\vec{r}}{dt}=\dfrac{dx}{dt}\hat{i}+\dfrac{dy}{dt}\hat{j}+\dfrac{dz}{dt}\hat{k}$

$$\vec{v}=ay\hat{i}+(-ax)\hat{j}+0\hat{k}$$

$$\frac{dx}{dt}=ay \text{ ………….…….(1)}$$

$$\frac{dy}{dt}=-ax \text{ …………...(2)}$$

$$\frac{dz}{dt}=0 \text{ ……………….(3)}$$

Differentiating (1) w.r. to t,

$$\frac{d^2x}{dt^2}=a\frac{dy}{dt} \text{ …………..(4)}$$

Substituting (2) into (4)

$$\frac{d^2x}{dt^2}=a(-ax)$$

$$\frac{d^2x}{dt^2}+a^2x=0$$

To find the solution of ODE:

$$m^2 + a^2 = 0$$
$$m = \pm ai$$
$$\therefore x(t) = A\cos at + B\sin at \;\ldots\ldots\ldots\ldots(5)$$

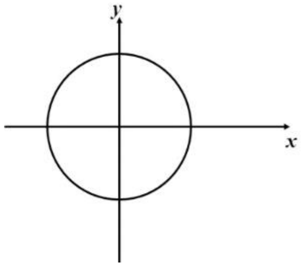
From (1) ,

$$y = \frac{1}{a}\frac{dx}{dt}$$
$$y = \frac{1}{a}\left[-A\sin at(a) + B\cos at(a)\right]$$
$$\therefore y(t) = -A\sin at + B\cos at \;\ldots\ldots\ldots..(6)$$

To plot

$$x^2 = A^2\cos^2 at + B^2\sin^2 at + 2AB\cos at\sin at$$
$$y^2 = A^2\sin^2 at + B^2\cos^2 at - 2AB\cos at\sin at$$
$$x^2 + y^2 = A^2(\cos^2 at + \sin^2 at) + B^2(\cos^2 at + \sin^2 at)$$
$$x^2 + y^2 = A^2 + B^2$$

By plotting path lines,



Example 2:

Given velocity, $\vec{v} = (ay, -a(x - bt), 0)$, find particles path where a and b are constant

Solution:

…………………… solve it

2.3 DEFINITIONS

a) Two dimensional flow

- Flow conditions change along two dimensions only

- If x, y, z are coordinates of any point in the fluid, then all physical quantities (velocity, density, pressure etc.) associated with the fluid are independent of z. Thus u, v are functions of x, y and t and $w = 0$ for such a motion.

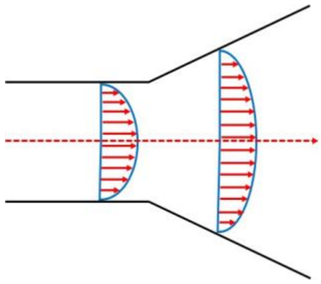


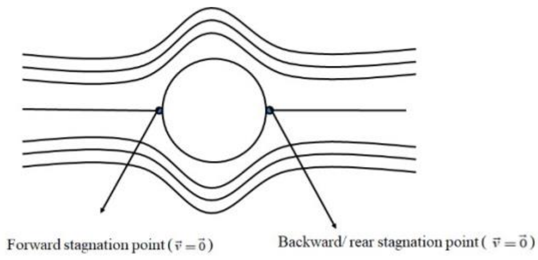
Fig 2.1 Two dimensional flow

b) Steady flow

- At any given point, the velocity does not vary with time in magnitude or direction.
- $\dfrac{\partial \vec{v}}{\partial t} = 0$

c) Stagnation point

- A point in a flow field where the local velocity of the fluid is zero.
- Stagnation points exist at the surface of objects in the flow field, where the fluid is brought to rest by the object



Example 3:

A steady incompressible, two dimensional flow of velocity are given as

a) $\vec{v} = (0.5 + 0.5x)\hat{i} + (1.5 - 0.8y)\hat{j}$

b) $\vec{v}=\left(1+\frac{10}{4}x+y\right)\hat{i}+\left(-\frac{1}{2}-\frac{3}{2}x-\frac{10}{4}y\right)\hat{j}$

Find the stagnation point of this flow field

Solution:

a) $\vec{v}=\vec{0} \qquad \Rightarrow \qquad (0.5+0.5x)\hat{i}+(1.5-0.8y)\hat{j}=0$

$0.5+0.5x=0 \qquad\qquad 1.5-0.8y=0$

$x=-1 \qquad\qquad y=1.875$

So, $(-1,1.875)$ is the stagnation point

b) …………… solve it

2.3.1 Method of describing fluid motion

a) Eulerian description

- The velocities are given at fixed point in a space $\vec{v}=(\hat{r},t)$ at time varies
- In this approach, individual fluid particles are considered and their motion tracked:

  Position of fluid particle:

  $(\vec{r},t)=x\hat{i}+y\hat{j}+z\hat{k}$ with x,y,z as coordinates and $\hat{i},\hat{j},\hat{k}$ as unit vectors

  Velocity of fluid particle:

  $(\vec{v},t)=\frac{d\vec{r}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}$ with u,v,w are velocity components in x,y,z coordinates

b) Lagrangian description

- Quantities are given for a fixed particle at different time
- So, the velocity is $\vec{v}=(r_0,t)$ where r_0 is the particle position at $t=0$

- In this approach, we are focusing on a certain point in space and consider the motion of fluid particles passing that point as time goes by:

  Position in space: $P = \left[x, y, z\right]$

  Velocity at point P: $u = f_1(x, y, z, t)$

  $$v = f_2(x, y, z, t)$$
  $$w = f_3(x, y, z, t)$$

Example 4:

In a Lagrangian description, the fluid particle at time, $t = 0$ is $(x_0 + y_0 t^2, y_0 + t, z_0)$

a) Find the Eulerian velocity field

b) Using velocity form a) if $\dfrac{D\vec{v}}{Dt} = \dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$, find $\dfrac{D\vec{v}}{Dt}$

c) Show that $\dfrac{D\vec{v}}{Dt}$ is the same as the Lagrangian acceleration when $t = 0$

Solution:

a) For fluid particle,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$$
$$\Rightarrow \quad x = x_0 + y_0 t^2$$
$$y = y_0 + t$$
$$z = z_0$$

Since $\vec{v}(\vec{r}, t) = \dfrac{d\vec{r}}{dt}$ then,

$$\frac{dx}{dt} = 2y_0 t$$
$$\frac{dy}{dt} = 1$$
$$\frac{dz}{dt} = 0$$

$\Rightarrow$ The Lagrangian velocity is $\vec{v}(\vec{r}_0, t) = 2y_0 t\hat{i} + \hat{j}$

$\Rightarrow$ The Eulerian velocity field, $\vec{v} = 2t(y-t)\hat{i} + \hat{j}$

b) $\vec{v} = 2t(y-t)\hat{i} + \hat{j}$

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$
$$\frac{\partial\vec{v}}{\partial t} = (2y-4t)\hat{i}$$

$$\vec{v}\cdot\nabla\vec{v} = \left[2t(y-t)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\left[2t(y-t)\hat{i} + \hat{j}\right]$$

$$= 2t(y-t)\left[\frac{\partial}{\partial x}(2t(y-t)) + \frac{\partial}{\partial x}(1)\right] + \frac{\partial}{\partial y}(2t(y-t)) + \frac{\partial}{\partial y}(1)$$
$$= 2t\hat{i}$$

$$\frac{D\vec{v}}{Dt} = (2y-4t)\hat{i} + 2t\hat{i}$$

$$= (2y-2t)\hat{i}$$

c) When $t = 0$, $\frac{D\vec{v}}{Dt}(r_0, t) = 2y_0\hat{i}$

Lagrangian acceleration, $\vec{a}(r_0, t) = \frac{D\vec{v}}{Dt}(r_0, t) = \frac{\partial}{\partial t}(2y_0 t)\hat{i}$

$$\vec{a} = 2y_0\hat{i}$$

Hence, proved.

2.4 STREAMLINES AND STREAKLINES

Streamlines are family of curve that are instantaneously tangent to the velocity vector of the flow.

Streaklines are the locus point for a fluid particle that are pass continuously through the particles spatial point in the pass.

- By definition, different streamlines at the same instant in a flow do not intersect because of fluid particles cannot have to different velocity at the same point
- Similarly, streaklines cannot intersect themselves or other streaklines because two particles cannot be present at the same location at the same instant of time
- Streamlines, streaklines and pathlines are identical in steady flow. In general streamlines, streak line and path lines are not the same. However, they coincide when the flow is steady
- Streamlines is the curve that is everywhere tangent to the instantaneous local velocity vector
- If $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ and velocity vector, $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$, then the equation of streamlines is $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$. This shows that the curves are parallel to the velocity vector
- Similar to pathlines, calculation of streamlines need to consider initial value when $x = x_0$, $y = y_0$ and $z = z_0$.

Example 5:

Given $\vec{v} = (ay, -a(x - bt))$. Find streamlines

Solution:

$u = ay, \qquad v = -a(x - bt)$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$
$$\frac{dx}{ay} = \frac{dy}{-a(x-bt)}$$

$$\int (-x + bt)\ dx = \int y\ dy$$

$$-\frac{x^2}{2} + btx = \frac{y^2}{2} + D$$
$$x^2 + y^2 - 2btx = D$$
$$(x - bt)^2 + y^2 = D + (bt)^2, D \text{ is constant}$$

Example 6:

Find the pathlines and streamlines if $\vec{v} = (2\cos \pi t, 2\sin \pi t, 0)$

Solution:

<u>Pathlines:</u>

$$\frac{dx}{dt} = 2\cos \pi t \ ....(1)$$
$$\frac{dy}{dt} = 2\sin \pi t \ ....(2)$$
$$\frac{dz}{dt} = 0$$

From (1) $\qquad \int dx = 2\int \cos \pi t\ dt$

$$x = \frac{2\sin \pi t}{\pi} + c_1$$
$$(x - c_1) = \frac{2}{\pi}\sin \pi t$$

From (2) $\qquad \int dy = 2\int \sin \pi t\ dt$

$$y=-\frac{2\cos\pi t}{\pi}+c_2$$

$$(y-c_2)=-\frac{2}{\pi}\cos\pi t$$

$$\Rightarrow (x-c_1)^2+(y-c_2)^2=\frac{4}{\pi^2},\ c_1 \text{ and } c_2 \text{ are constant}$$

<u>Streamlines:</u>

$$\frac{dx}{2\cos\pi t}=\frac{dy}{2\sin\pi t}$$

$$\frac{\sin\pi t}{\cos\pi t}dx=dy$$

$$\int\tan\pi t dx=\int dy$$

$$x\tan\pi t=y$$

$$\Rightarrow y=x\tan\pi t$$

Example 7:

Find pathlines and streamlines if $\vec{v}=(a(t)x,-a(t)y,0)$ where $a(t)$ is scalar function of t

Solution:

…………… solve it

Practice problems

1. For a two dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u=x+y+2t$ and $v=2y+t$.
   Determine the Lagrangian coordinates as function of the initial positions x_0, y_0 and the time t.
2. If the velocity distribution is $\vec{q}=Ax^2y\hat{i}+By^2zt\hat{j}+Czt^2\hat{k}$ where A,B,C are constants, then find the acceleration and velocity component.

3. The velocity components of a flow in a cylindrical polar coordinates are $(r^2 z\cos\theta, rz\sin\theta, z^2 t)$. Determine the components of the acceleration of a fluid particle.
4. Obtain the streamlines of a flow $u = x,\ v = -y$
5. The velocity field at a point in fluid is given as $\vec{q} = \left(\frac{x}{t}, y, 0\right)$. Find pathlines and streamlines.
6. Find the streamlines and path of the particle when

   $u = \frac{x}{1+t};\ v = \frac{y}{1-t};\ w = \frac{z}{1+t}$.