CHAPTER 3

MASS CONVERSATION AND STREAM FUNCTIONS

3.1 CONTINUITY EQUATION

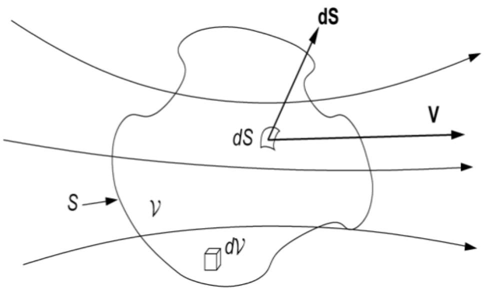


Fig 3.1 Finite control volume fixed in space

- Continuity equation is a statement of mass conservation converted into a differential equation
- From the Fig 3.1, rate of flow of mass into V , given as

$$-\int_S \rho\vec{v}\cdot dS, \qquad \rho = \rho(x,y,z,t)$$

where $\rho\vec{v}$ is a mass flux with density, ρ

- This integral measure the total fluid across the surface S per unit time
- Also rate of increases of mass into V , given as

$$\frac{d}{dt}\int_V \rho \ dV$$

And since V is a fix region this can be return as $\int_V \frac{d\rho}{dt}\ dV$

Due to mass is conserve, we must have

$$\int_V \frac{d\rho}{dt}\ dV = -\int_S \rho\vec{v}\cdot dS \qquad \text{………………..(1)}$$

Knowing the divergence theorem is

$$\int_V \nabla\cdot\vec{A}\ dV = \int_S \vec{A}\cdot dS$$

so,

$$\int_S \rho\vec{v}\cdot dS = \int_V \nabla\cdot\rho\vec{v}\ dV \qquad \text{……………….(2)}$$

Substitute (2) into (1)

$$\int_V \frac{d\rho}{dt}\ dV = -\int_V \nabla\cdot\rho\vec{v}\ dV$$

$$\int_V \left[\frac{d\rho}{dt} + \nabla\cdot\rho\vec{v}\right] dV = 0 \qquad \text{………………..(3)}$$

Then from (3)

$$\frac{d\rho}{dt} + \nabla\cdot\rho V = 0 \qquad \text{………………..(4)}$$

Equation (4) is called continuity equation or mass conservation equation.

Notes:

1. If ρ is constant, then the continuity equation is $\nabla\cdot\vec{v} = 0$ (incompressible)
2. If ρ in independent of t, then the continuity equation is $\nabla\cdot\rho\vec{v} = 0$ (compressible)

3.2 THE CONVECTIVE DERIVATIVE

- In the case of incompressible fluid, ρ does not depend on $\vec{v}$, but each of the particles keep its density
- Suppose a particle of one position, $\vec{r}$ and time t has density of $\rho(\vec{r},t)$ when it moves and when $\vec{r}+\vec{v}\delta t$ and $t+\delta t$ the density must become $\rho(\vec{r}+\vec{v}\delta t, t+\delta t)$. So, clearly a change on density, ρ is

$$\rho(\vec{r}+\vec{v}\delta t, t+\delta t) - \rho(\vec{r},t)$$

If we consider

$$\lim_{\delta t\to 0} \frac{\rho(\vec{r}+\vec{v}\delta t, t+\delta t) - \rho(\vec{r},t)}{\delta t} = \vec{v}\cdot\nabla\rho + \frac{\partial\rho}{\partial t} \qquad \text{(Taylor's theorem)}$$

Or we write

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho \quad \text{..................(5)}$$

Equation (5) is known as **convective derivative**

Using vector identity $\nabla\cdot\phi\vec{A} = \vec{A}\cdot\nabla\phi + \phi\nabla\cdot\vec{A}$ into (5), then

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} - \rho\nabla\cdot\vec{v}$$

or

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0 \ \text{ since } \ \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$$

3.4 THE STREAM FUNCTION FOR 2D FLOW

- From the continuity equation for incompressible fluid $\nabla\cdot\vec{v} = 0$ means by using divergence theorem that is no total volume through and closed surface as much flows out as flow in.
- For the same of mathematics simply we choose to investigate the flow in which there are two non-zero velocity and two effective coordinates.
- There are:
  i. Two-dimensional flow where the velocity defines as

  $$\vec{v} = u(x,y)\hat{i} + v(x,y)\hat{j}$$

  Or in polar plane

  $$\vec{v} = V_r(r,\theta)\hat{r} + V_\theta(r,\theta)\hat{\theta}$$

  ii. Asymmetric flow (no swirl)

  $$\vec{v} = V_r(r,z)\hat{r} + V_\theta(r,z)\hat{\theta}$$

  Which shows that velocity not depend on θ

iii. In cylindrical coordinate, vector $\vec{v}$ is independent of θ and there is no θ velocity. Such flow has two velocity components V_r, V_z and one differential equation $\nabla \cdot \vec{v} = 0$ connecting them

$$\vec{v} = V_r(r,z)\hat{r} + V_z(r,z)\hat{k}$$

4.3.1 Existence of Stream Function

- In 2D flow of incompressible fluid

$$\vec{v} = u(x,y)\hat{i} + v(x,y)\hat{j}$$

Continuity equation is given as

$$\nabla \cdot \vec{v} = 0 \qquad \text{……………(9)}$$

Equation (9) is fully satisfied equations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \qquad \text{……………(10),(11)}$$

For any suitable differential function $\psi(x,y)$. Function $\psi(x,y)$ is known as **stream function.**

To proof:

$$\begin{aligned}\nabla \cdot \vec{v} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \\ &= 0\end{aligned}$$

To show that $\psi(x,y)$ exist and can be written in terms u and v, integrate $u = \frac{\partial \psi}{\partial y}$ with respect to y and let $y = N$ $(b \le N \le y)$ where b is constant, gives

$$\psi = \int_b^y U(x,N)dN + \alpha(x) \qquad \text{……………..(12)}$$

where $x(t)$ is a function of x and b is constant.

Differentiate (12) with respect to x

$$\frac{\partial\psi}{\partial x}=\int_b^y \frac{\partial}{\partial x}U(x,N)dN+\alpha'(x) \qquad \ldots\ldots\ldots\ldots(13)$$

From (9)

$$\nabla\cdot\vec{v}=0$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial N}=0$$
$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial N}$$

so from (13)

$$\frac{\partial\psi}{\partial x}=\int_b^y -\frac{\partial\psi}{\partial N}dN+\alpha'(x)$$
$$=-v(x,y)+v(x,b)+\alpha'(x)$$

Using (11)

$$-v(x,y)=-v(x,y)+v(x,b)+\alpha'(x)$$
$$\alpha'(x)=-v(a,b)$$

Integrate $\alpha'(x)$ with respect to x and let $x=M$ $(a\le M\le x)$

$$\alpha(x)=-\int_a^M v(M,b)\ dM \text{ where } a \text{ is constant}$$

Then from (12),

$\varphi=\int_b^y u(x,N)dN-\int_a^x v(M,b)dM$. It shows that $\psi(x,y)$ exist in terms of u and v for any constants a and b.

- Determination of $u=\dfrac{\partial\psi}{\partial y}$ and $v=-\dfrac{\partial\psi}{\partial x}$

  - If $\nabla\cdot\vec{B}=0$, so there exists a vector $\vec{A}$ such that $\vec{B}=\nabla\times\vec{A}$, $\vec{A}$ is called the vector potential of $\vec{B}$.

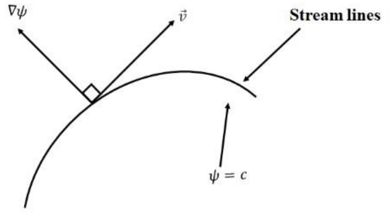
- By using definition, we can say that $\nabla \cdot \vec{v} = 0$, there exist $\vec{A}$ such that $\vec{v} = \nabla \times \vec{A}$
- If we write $\vec{A} = \psi(x, y)\hat{k}$ then

$$\vec{v} = \nabla \times \vec{A}$$
$$=$$
$$=$$

$$\Rightarrow u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

4.4 PROPERTIES OF STREAM FUNCTION

Let $\vec{v} = \nabla \psi + \hat{k}$



Vector, $\vec{v}$ is perpendicular to $\nabla \psi$ and $\nabla \psi$ is perpendicular to $\psi = c$ where c is constant. Therefore, vector $\vec{v}$ is parallel to the curve $\psi = c$. Curve $\psi = c$ are streamlines.

Example 1:

Given $\psi(x, y) = Uy$, where U is a constant. Sketch the streamlines.

4.5 SOME BASIC STREAM FUNCTIONS

a) Flows parallel to the x-axis

i. *Uniform stream of speed U parallel to the x-axis*

Let $\vec{v} = U\hat{i} + 0\hat{j}$, U is constant. Find $\psi(x, y)$

Solution:

ii. *Uniform shear flow*

Let $\vec{v} = \beta y\hat{i} + 0\hat{j}$, β is constant. Find $\psi(x, y)$

iii. *First flow near a wall in channel*

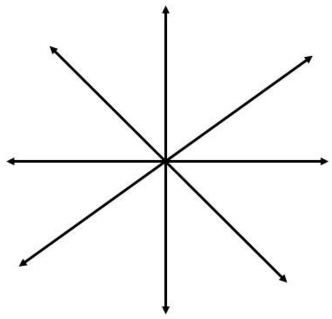
Let $\vec{v} = U\ln\left(\frac{y}{a}\right)\hat{i} + 0\hat{j}$ where U and a are constant. Find $\psi(x, y)$

iv. *Transition layer*

Let $\vec{v} = \left(U_0 + \frac{1}{2}(U_1 - U_0)\left[1 + \tanh\left(\frac{y}{a}\right)\right]\right)\hat{i} + 0\hat{j}$ where U_0, U_1 and a are constant. Find $\psi(x, y)$.

b) Flow radially outwards

Flow leaves the vicinity/ source radially with the same speed at all angles .



Given $\vec{v} = f(r)\hat{r} + 0\hat{\theta}$. We need to consider stream function in plane polar coordinates.

In polar coordinates,

$$\vec{v} = V_r\hat{r} + V_\theta\hat{\theta}$$

To show the relationship between velocity, v and stream function .. in polar coordinates

$$\vec{v} = \nabla\psi \times \hat{k}$$

We need to use cylindrical polar coordinates, Given $\vec{v} = f(r)\hat{r} + 0\hat{\theta}$. Find $\psi(x, y)$

$$\frac{\partial \psi}{\partial r} = 0 \qquad \text{..................(1)}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = f(r) \text{ ....(2)}$$

We can show that

$$\psi = rf(r)\theta \quad \text{....................(3)}$$

But to satisfy (1), $\psi(r, \theta)$ is independent of r and $\frac{\partial \psi}{\partial \theta}$ must be independent of θ.

Let $\quad \frac{\partial \psi}{\partial \theta} = A$

Since

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$V_r = \frac{1}{r}A$$

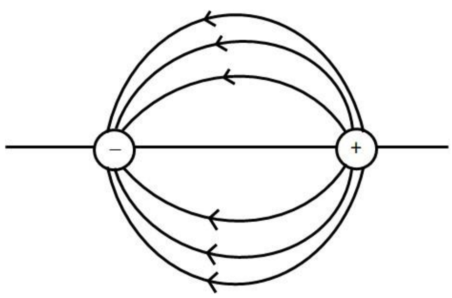
$$\therefore f(r) = \frac{1}{r}A$$

From (3),

$$\psi = r\left(\frac{1}{r}A\right)\theta$$

$$= A\theta$$

c) A dipole along the x-axis



Given velocity

$$\vec{v}(r,\theta) = 0\hat{r} + V_\theta(r,\theta)\hat{\theta}$$

The appropriate stream function is $\psi(r,\theta) = \dfrac{\mu}{r}\sin\theta$ where μ is a constant

d) Fluid in circle

Given velocity

$$\vec{v}(r,\theta) = 0\hat{r} + f(r)\hat{\theta}$$

There are two special cases of $f(r)$:

i. $V_\theta = f(r) = \dfrac{c}{r}$ (line vortex)

ii. $V_\theta = f(r) = Dr$ (rigid body motion)

We know that,

$$V_\theta = -\frac{\partial\psi}{\partial r}$$

$$\vec{v} = V_r\hat{r} + V_\theta\hat{\theta}$$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$V_\theta = -\frac{\partial\psi}{\partial r}$$

$$-\frac{\partial\psi}{\partial r} = Dr$$

$$\psi = -\frac{Dr^2}{2} + A(\theta)$$

From (i),

$$-\frac{\partial\psi}{\partial r} = \frac{c}{r}$$

$$\psi = -c\ln\left(\frac{r}{a}\right) \quad ; \quad c = \frac{k}{2\pi}, \text{ where } k \text{ is circulation}$$

Therefore line vortex has stream function, $\psi = -\dfrac{k}{2\pi}\ln\left(\dfrac{r}{a}\right)$.

3.6 THE STREAM FUNCTION FOR AXISYMMETRIC FLOW (STOKES)

In axisymmetric flow,

$$\vec{v}(r,z) = U_r(r,z)\hat{r} + V_z(r,z)\hat{k}$$

a) Existence of Stokes stream function, $\psi(r,z)$

Hence, we used cylindrical coordinates by taking

$$\vec{v}(r,z) = U_r(r,z)\hat{r} + V_z(r,z)\hat{k}$$

And using continuity equation, $\nabla \cdot \vec{v} = 0$

We may consider, $\frac{\partial}{\partial r}(rU_r) + \frac{\partial}{\partial z}(rV_z) = 0$

Therefore, it can be shown that the stream function in terms of U_r and V_z is

b) Determination of axisymmetric stream function in cylindrical and spherical coordinate

Since $\nabla \cdot \vec{v} = 0$, then axis vector $\vec{A}$ such that $\vec{v} = \nabla \times \vec{A}$ where $\vec{A}$ is vector potential.

Let $\vec{A} = \frac{1}{r}\psi\theta$.

Therefore, by using cylindrical coordinates, In axisymmetric flow, $\vec{v} = V_r(r,z)\hat{r} + V_z(r,z)\hat{k}$

$$\therefore V_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$\therefore V_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

Also, since $\nabla \cdot \vec{v} = 0$, then there exist $\vec{A}$ such that $\vec{v} = \nabla \times \vec{A}$ where $\vec{A}$ is vector potential

$$\vec{A} = \frac{1}{r\sin\theta}\psi\hat{\lambda}$$

Therefore, in spherical coordinates,

In spherical we have,

$$\therefore V_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$$

$$\therefore V_z = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$