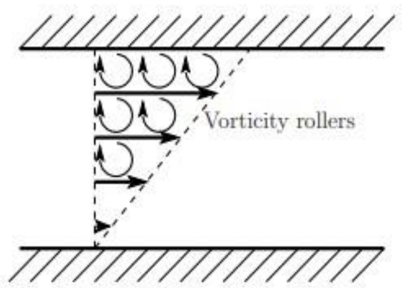
CHAPTER 4

VORTICITY

4.1 DEFINITTION

- Vorticity is defined as $\vec{w} = \nabla \times \vec{v}$. It means that rotation rate of flow as an angular velocity along the direction of vorticity



- If $\vec{w} = 0$, the flow is irrotational

Example 1:

Given $\vec{v} = \frac{y}{x^2 + y^2}\hat{i} - \frac{x}{x^2 + y^2}\hat{j}$

i. Find vorticity $\vec{w}$
ii. Find streamlines

Solutions:

i. Vorticity

ii. Streamlines

$\Rightarrow$ From this example, it appears that the fluid is irrotational, but if we observe an individual particle each of that particles retain its orientation with respect to coordinates axis as it moves in a circle, $\vec{w}$ is actually measure local rotation.

$\Rightarrow$ If the fluid is rotating as each were rigid at angular velocity, $\vec{\Omega}$ then $\vec{w}=2\vec{\Omega}$ or if $\vec{w}\neq 0$ each fluid particles is rotating at angular velocity $\vec{\Omega}=\frac{1}{2}\vec{w}$.

Theorem

If $\nabla\times\vec{v}=0$, then there exists a scalar ϕ, such that $\vec{v}=\nabla\phi$

Proof

$\nabla\times\nabla\phi=0$ (… Show it)

Since, $\nabla\cdot\vec{v}=0$, if $\vec{v}=\nabla\phi$

$\Rightarrow$ $\nabla\cdot\vec{v}=\nabla\cdot\nabla\phi=\nabla^2\phi$

Show also, from the above theorem, $\nabla^2\psi=0$ (… Show it)

4.2 THE RELATIONSHIP BETWEEN STREAM FUNCTION AND VORTICITY

a) 2D Flow

Consider $\vec{v} = \nabla \times \psi \hat{k}$

We know that $\vec{w} = \nabla \times \vec{v}$

$$= \nabla \times \left(\nabla \times \psi \hat{k} \right)$$

And by using vector identity, we have

$$\vec{w} = \nabla \left(\nabla \cdot \psi \hat{k} \right) - \nabla^2 \psi \hat{k}$$

But, $\nabla \cdot \psi \hat{k} = 0$

Therefore, $\vec{w} = -\nabla^2 \psi \hat{k}$

b) Axisymmetric Flow

i. Cylindrical coordinates

We know that $\vec{v} = \left[-\frac{1}{r} \frac{\partial \psi}{\partial z}, 0, \frac{1}{r} \frac{\partial \psi}{\partial r} \right]$

Since $\vec{w} = \nabla \times \vec{v}$

Then, $\vec{w} = -\frac{1}{r} D^2 \psi \hat{\theta}$ where $D^2 = \frac{\partial^2}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)$

ii. Spherical coordinates

We know that $\vec{v} = \left[-\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}, 0 \right]$

Since $\vec{w} = \nabla \times \vec{v}$

Then, $\vec{w} = \frac{1}{r \sin\theta} D^2 \psi \hat{\lambda}$

where $D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \theta^2} - \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial \theta} \right)$

Example 2:

Flow with non-zero vorticity. Given $\vec{v} = u(y)\hat{i}$. Find $\vec{w}$

Solution:

Example 3:

Flow in circles round the *z-axis*. Given $V_r = 0$ and $V_\theta = f(r)$ Find $\vec{w}$ if

a) $V_\theta = Dr$

b) $V_\theta = \dfrac{c}{r}$

Solution:

a)

b)

c) **Model with Vortices**

- We only consider 2D vortex: **Rankine's vortex**
- This model is chosen because most of all line vertex, $\vec{w}=0$, but only near $r=0$, (r is radius of a circle).
- For example, if $V_\theta = \frac{c}{r}$ means when $r \to \infty, V_\theta \to 0$ but when $r \to 0, V_\theta$ undefined

Example 4:

Consider

$$\vec{w} = \begin{cases} \Omega \hat{k} & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

where a is a radius of a circle. Find $\vec{w}(r)=0$ on $r=a$ and $\psi(r)$ bounded when $r \to 0$

Solution:

Using boundary condition: $\psi(r)=0$ on $r=a$

$\psi(r)$ is bounded when $r\rightarrow 0$

Let $A=0$ ($\ln 0$: undefined)

Case $r>a$;

4.3 CIRCULATION

- Let c be any curved in a fluid region. The circulation of fluid around the curve is defined as

$$\Gamma = \oint_c \vec{v} \cdot d\vec{r} \quad \text{where } d\underline{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Theorem

If c is reducible within region of flow in which $\vec{w} = 0$ then $\Gamma = 0$ (irrotationally ~ zero circulation)

Example 5:

Find Γ if $\vec{v} = \dfrac{y}{x^2 + y^2}\hat{i} - \dfrac{x}{x^2 + y^2}\hat{j}$

When c is a curve of circle

Solution:

4.4 LAPLACE EQUATION

Mostly all fluids are incompressible and satisfy the continuity equation

$$\nabla \cdot \vec{v} = 0$$

If the flow is irrotational, $\nabla \times \vec{v} = 0$

Then $\vec{v} = \nabla \phi$

Therefore, from the continuity equation

$$\nabla \cdot \vec{v} = 0$$
$$\nabla \cdot \nabla \phi = 0$$

$\nabla^2 \phi = 0$ (is known as **Laplace equation**)

In cartesian coordinate,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

In cylindrical coordinate,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

In spherical coordinate,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 \phi}{\partial \lambda^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\cot\frac{\partial^2 \phi}{\partial \theta^2}$$

Example 6:

The solution of $\nabla^2 \phi = 0$ in spherical coordinate is given as

$$\phi(r,\theta) = \left(Ar^n + Br^{-n-1}\right)\left(CP_n \cos\theta + DP_n \sin\theta\right)$$

By taking $n = 1,\ C = 1,\ D = 0,\ P_n \cos\theta = \cos\theta$

and consider the flow is irrotational when $r \to \infty$

Show that $\vec{v} = A\hat{k}$

Solution: