CHAPTER 5
THE EQUATION OF MOTION

5.1 REAL FLUID AND THE EFFECT OF VISCOSITY

- Real fluids and viscos are synonymous term where all real fluid are more or less viscos
- Real fluid tend exert forces tangent to the surfaces with which they are in contact
- In real fluid, tangential or shearing forces always causes flow to fluid function because this viscos forces oppose the sliding one of particle pass another. This friction forces are due to property call viscosity
- By definition, viscosity is the measure of the fluid resistant to deformation

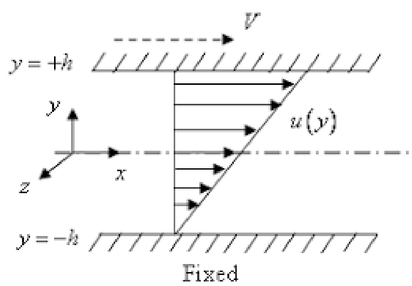


Fig 5.1

- From Fig 5.1, both plates are infinite in x and y directions.
- Fix plate occupies the whole plane $y = 0$ and in moving plate occupies the whole plane $y = h$
- The upper plate moves with a constant speed $u(y)$ in the positive x direction
- Fluid lies in the region $0 < y < h$

  The behaviour of the fluid in this region is important to investigate

Case I: Inviscid Fluid (fluid with no resistance)

In this region, nothing is happened. The upper plate slips freely over the top of an inviscid fluid

Case II: Real Fluid (Viscous fluid)

From experiment, the top layer of fluid moves with the top plate. Whilst the bottom layer stays fixed with the bottom plate.

One layer of fluid influences each neighbor so that the one final configuration involved a smooth transilation from 0 to u as y goes from 0 to h

5.1.1 THE TANGENTIAL STRESS (SHEAR STRESS), $\tau(y)$

- This stress required to move the top plate
- Experiments have shown that the force, F is required to move the plate is proportional to U and inversely proportional to the distance, h and also proportional to the area of the plate

$$\left.\begin{array}{r} F \propto u \\ F \propto \dfrac{1}{h} \\ F \propto \text{area of plate} \end{array}\right\} \frac{F}{\text{unit area}} = \tau = \mu \frac{u}{h}$$

  where μ is constant of proportionality and known as the viscosity (or the coefficient of shear viscosity)

- τ is a force per unit area and known as the tangential stress
- Further experiment suggests that the gap between the moving plate is δy and two surfaces with velocity $u(y+\delta y)$ respectively and

$$\tau(y+\delta y)=\mu\left[\frac{u(y+\delta y)-u(y)}{\delta y}\right]$$

- By taking unit $\delta y \to 0, \quad \tau(y)=\mu\frac{\partial u}{\partial y}$

  This force is along any surfaces in the fluid perpendicular to y

5.1.2 THE SHEAR VISCOSITY, μ

- The shear viscosity, μ is also called the dynamic viscosity (Newtonian viscosity)
- The viscosity of the fluid is a measure of its resistant to shear or to angular deformation
- Dimension of viscosity $\frac{M}{LT}\left(\frac{kg}{ms}\right)$
- In many fluid problems, it occurs that the volume of viscosity divided by the density, ρ

  This called the kinematic viscosity and denoted by $\nu=\frac{\mu}{\rho}$

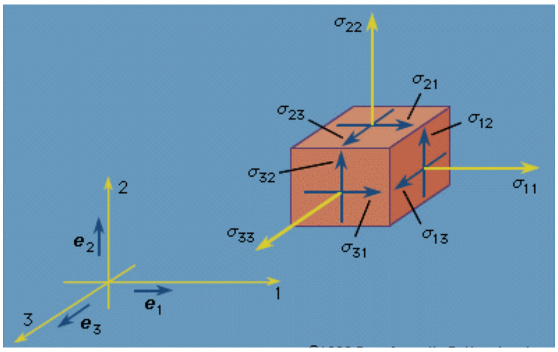
5.2 NAVIER-STOKES EQUATION

5.2.1 THE STRESS TENSOR AND CONSTITUTE EQUATION

- The force τ on a surface in the fluid with vector normal, η is given by

  $\tau_i=\sigma_{ij}\eta_j$ ....(1)

  in tensor notation, where σ_{ij} is stress tensor
- This stress tensor represents the forces per unit area in the in the i^{th} direction on a surface with normal in the j^{th} dimension



From (1),

$$\tau_i = \sigma_{i1}\eta_1 + \sigma_{i2}\eta_2 + \sigma_{i3}\eta_3 \quad \ldots\ldots\ldots(2)$$

For $i = 1, 2, 3$:

$$\tau_1 = \sigma_{11}\eta_1 + \sigma_{12}\eta_2 + \sigma_{13}\eta_3 \quad \ldots\ldots\ldots\ldots(3)$$
$$\tau_2 = \sigma_{21}\eta_1 + \sigma_{22}\eta_2 + \sigma_{23}\eta_3 \quad \ldots\ldots\ldots\ldots(4)$$
$$\tau_3 = \sigma_{31}\eta_1 + \sigma_{32}\eta_2 + \sigma_{33}\eta_3 \quad \ldots\ldots\ldots\ldots(5)$$

Generalisation for stress tensor.
Given that,

$$\sigma_{ij} = -p\delta_{ij} + \bar{\sigma}_{ij}$$

where $\bar{\sigma}_{ij}$ denoted the departure of σ_{ij} from inviscid value, p is pressure and $\bar{\sigma}_{ij}$ is defined as

$$\bar{\sigma}_{ij} \propto \frac{\partial U_i (i^{th} \text{ velocity})}{\partial x_j (j^{th} \text{ coordinate})}$$

or can be written as

$$\bar{\sigma}_{ij} = \mu \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right]$$

where velocity

$$\hat{\mu} = u\hat{i} + v\hat{j} + w\hat{k}, \qquad u = u_1, \;\; v = u_2, \;\; w = u_3$$

In this case, σ_{ij} is proven to be symmetric,

$\sigma_{ij} = \sigma_{ji}$ therefore $\bar{\sigma}_{ij} = \bar{\sigma}_{ji}$. Hence,

$$\sigma_{ij} = -p\delta_{ij} + \mu \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \quad \ldots\ldots\ldots\ldots(6)$$

(6) is known as constitute equation for Newtonian fluid

5.2.2 DERIVATION OF EQUATION OF MOTION

We start with Newton's Law, $F = ma$, which means total forces = $\sum ma$

$$\int_s \vec{\tau}\, ds + \int_v \rho F dv = \int_v \rho \frac{Du}{Dt}\, dv$$

where

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}$$

In tensor notation,

$$\int_s \tau_i\, ds + \int_v \rho F_i\, dv = \int_v \rho \frac{Du_i}{Dt}\, dv$$

or

$$\int_s \sigma_{ij}\eta_j\, ds + \int_v \rho F_i\, dv = \int_v \rho \frac{Du_i}{Dt}\, dv$$

Using Gauss Theorem for the first term, then

$$\int_s \frac{\partial \sigma_{ij}}{\partial x_i}\, dv + \int_v \rho F_i\, dv = \int_v \rho \frac{Du_i}{Dt}\, dv$$

and deduce this equation to obtain

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho F_i = \rho \frac{Du_i}{Dt} \quad \text{.............(7)}$$

(7) is the expression of motion for any fluid of Newtonian incompressible fluid

Using constitute equation (6) into (7)

$$\frac{\partial}{\partial x_i}\left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho F_i = \rho \frac{Du_i}{Dt} \quad \text{.............(8)}$$

where

$$\frac{\partial}{\partial x_i}\left(-p\delta_{ij} \right) = -\nabla p$$

$$i = 1,2,3 \qquad \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

$$\frac{\partial}{\partial x_1}\left(-p\delta_{1j}\right)=\frac{\partial}{\partial x_1}\left(-p\delta_{11}\right)$$
$$\frac{\partial}{\partial x_2}\left(-p\delta_{2j}\right)=\frac{\partial}{\partial x_2}\left(-p\delta_{22}\right)$$
$$\frac{\partial}{\partial x_3}\left(-p\delta_{3j}\right)=\frac{\partial}{\partial x_3}\left(-p\delta_{33}\right)$$

$$\begin{aligned}\Rightarrow \frac{\partial}{\partial x_i}\left(-p\delta_{ij}\right) &= \frac{\partial}{\partial x_1}(-p)+\frac{\partial}{\partial x_2}(-p)+\frac{\partial}{\partial x_3}(-p)\\ &= -p\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}\right)\\ &= -\nabla p\end{aligned}$$

$$\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]=\mu\nabla^2\vec{u}$$

After simplification in vector form, this equation can be written as

$$\frac{Du}{Dt}=-\frac{1}{p}\nabla p+\mu\nabla^2\vec{u}+F \quad ....(9)$$

Equation (9) known as **Navier-Stokes equation**