CHAPTER 6 (PART I)
SOLUTION OF NAVIER STOKES EQUATION

- The motion of incompressible fluid of constant density, ρ and constant kinematic viscosity, υ influenced by body force, g is governed by

$$\frac{\partial \vec{u}}{\partial t}+\left(\vec{u}\cdot\nabla\right)\vec{u}=-\frac{1}{\rho}\nabla p+\upsilon\nabla^2\vec{u}+\vec{g} \text{ ------------ (1)}$$

and continuity equation

$$\nabla\cdot\vec{u}=0 \text{ or } \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\right)$$

- Most of the fluid problems are solve with boundary condition in case of no slip condition
- No slip condition means the tangential components of velocity at a rigid boundary condition must be equal to those of boundary itself
- The normal components of velocity at a rigid boundary must also be equal to the boundary equation (1) can be written in each component as:

In x component:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x \text{ --------- (2)}$$

In y component:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y \text{ --------- (3)}$$

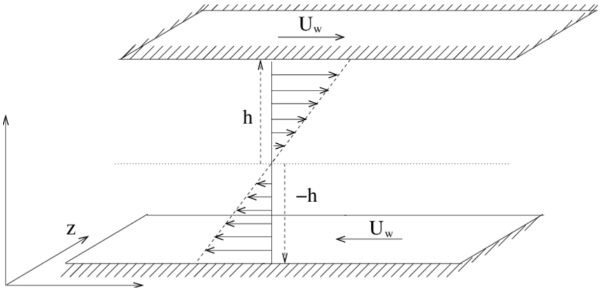
In z component:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z \text{ ------- (4)}$$

6.1 EXACT SOLUTION OF NAVIER STOKES EQUATION

a) Steady Plane Couette Flow

consider the steady flow between two parallel surfaces separated by distance h



The flow to be uni-directional flow in x-direction
So, there is no effect of gravity or $\vec{g} = 0$

Simplification:

1. Steady $\Rightarrow \frac{\partial}{\partial t} = 0$
2. Uni-directional flow in x-direction $\Rightarrow$ $v = w = 0$
3. $\vec{g} = 0$ $\Rightarrow$ $g_x = g_y = g_z = 0$

The continuity equations become

From Equation (2),

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon \frac{\partial^2 u}{\partial y^2} \text{ -------------- (5)}$$

From Equation (3),

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

From Equation (4): Not exist

<u>Case I: No pressure gradient and the bottom plate is stationary while top plate is moving with a uniform speed,</u> u

From Equation (5),

$$\upsilon \frac{\partial^2 u}{\partial y^2} = 0$$

Boundary conditions:

$$u = 0 \quad \text{at} \quad y = 0$$
$$u = u \quad \text{at} \quad y = h$$

Find A and B using BCs,

Case II: Pressure gradient exist and both plates are stationary

From Equation (5)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2}$$

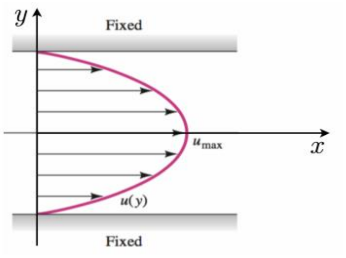
Let $$\frac{\partial p}{\partial x} = P_x$$

Boundary conditions:

$$u = 0 \text{ at } y = 0$$
$$u = 0 \text{ at } y = h$$

Find A and B using BCs,

Plot using BCs



The result shows that we required $P_x < 0$ for the flow to exist. It seems that when the plate is not moving, the flow is driven by a pressure gradient. Max value of velocity occurred at $y = \frac{h}{2}$

Case III: Pressure gradient exist and top plate is moving while bottom plate is stationary

The solution is the same as (6) where BCs,

$$u = 0 \text{ at } y = 0$$
$$u = u \text{ at } y = h$$

Using BCs the solution is

$$u(y) = \frac{P_x}{2\mu} y(y-h) + \frac{\mu}{h} y$$

b) Unsteady Couette Flow (Couette-Poissule Flow)

We consider two-dimensional flow moving in x-direction (uni-directional)

Simplification:

1. $\vec{g} = 0$
2. $v = w = 0$
3. $u = u(x, y, t)$

Case I: No pressure gradient and the lower plate is moving with speed, u

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial u}{\partial x} = 0$$
$$u = u(y, t)$$

Momentum equation

From Equation (2)

Equation (3) and (4) eliminate: no pressure gradient

Boundary conditions:

$$\begin{aligned} t=0: &\quad u=0, \quad 0\le y\le h \\ t\ge 0: &\quad u=u, \quad y=0 \\ &\quad u=0, \quad y=h \end{aligned}$$

Need to solve (7) by using separation of variables and applied the BCs, the solution is

$$u(y,t)=u\left(1-\frac{y}{h}\right)-\frac{2u}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\upsilon\left(\frac{n\pi}{h}\right)^2 t}\sin\frac{n\pi y}{h}$$

(7) is also can be solved by using similarity transformation

Given

$$u(y,t)=f(\eta), \quad \eta=\frac{y}{(\upsilon t)^{\frac{1}{2}}} \text{ --------------- (8)}$$

By using (8) into (7), the equation is reduced to

Substitute (i) and (ii) into (7)

$$f''+\frac{\eta}{2}f'=0 \text{ ----------- (9)}$$

Similarity transformation is used for reduce PDE (7) into ODE (9)

Boundary conditions

Since $u\left(y,t\right)=f\left(\eta\right)$ and $\eta=\frac{y}{\left(\upsilon t\right)^{\frac{1}{2}}}$

Solve (9) using reduction of order method

Then,

$$f(\eta) = -\frac{u}{\sqrt{\pi}} \int_0^{\eta} e^{-\frac{1}{4}s^2} ds + u$$

Since $u(y,t) = f(\eta)$, the solution is

$$u(y,t) = -\frac{u}{\sqrt{\pi}} \int_0^{\eta} e^{-\frac{1}{4}s^2} ds + u$$