CHAPTER 6 (PART II)
SOLUTION OF NAVIER STOKES EQUATION

6.1 EXACT SOLUTION OF NAVIER STOKES EQUATION

c) Unsteady Flow due to Oscillating Flow

Consider also two-dimensional flow moving in x-direction (uni-directional flow)

The general solution is obtained similar to problem **(b)** as

$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial y^2} \text{ -------------- (i)}$$

but the boundary conditions become

$$u(0,t) = u\cos\omega t \qquad \text{----------- (ii)}$$

$$u(\infty,t) \text{ is bounded ----------- (iii)}$$

To solve, need to consider method of separation of variables by taking $u(y,t) = e^{i\omega t} f(y)$

and substitute into (i)

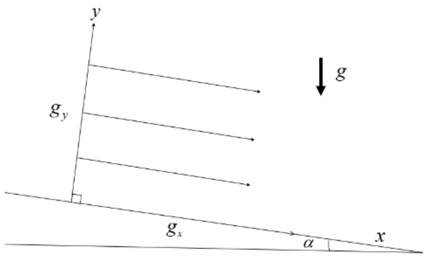
Second order homogeneous equation. Take $f = e^{m\eta}$

$$u(y,t) = e^{i\omega t}\left[Ae^{\sqrt{\frac{\omega}{2\upsilon}}(1+i)y} + Be^{-\sqrt{\frac{\omega}{2\upsilon}}(1+i)y} \right] \text{ ------------ (iv)}$$

After using (2) and (3), the solution is
u is bounded if $\mathrm{y} \to \infty$.

$u = u\cos\omega t$, $y = 0$:

d) Steady Flow under Gravity Down on Inclined Plane



$$\sin\alpha = \frac{g_x}{g} \qquad \cos\alpha = \frac{g_y}{g}$$

Consider the fluid as two-dimensional incompressible uni-directional flow.

Simplification:

1. Steady - not depend on t
2. 2D - depend on x and y
   - $u(x,y)$ and $v(x,y)$
3. Uni-directional flow $v(x,y)=0$

From continuity equation:

$$\Rightarrow u = u(y)$$

From Equation (2) (Chapter 6 Part I)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} + g_x \text{ -------------- (i)}$$

From Equation (3) (Chapter 6 Part I)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y \text{ -------------- (ii)}$$

Therefore, since $g_y = g\cos\alpha$

$$\frac{\partial p}{\partial y} = \rho g\cos\alpha$$

$$p(x,y) = y\rho g\cos\alpha + f(x) \text{ -------------- (iii)}$$

Equation (4) (Chapter 6 Part I) fully eliminated.

Note:

From (iii)

From (i)

Therefore,

Boundary conditions

$$u = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = h$$

Find A and B using BCs,

Then the solution becomes

e) Flow in a Pipe

Navier-Stokes equation in cylindrical coordinate is given as continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right)+\frac{1}{r}\frac{\partial U_\theta}{\partial \theta}+\frac{\partial U_z}{\partial z}=0 \quad \text{-------------- (i)}$$

where $\vec{v}=\left(U_r,U_\theta,U_z\right)$

Momentum equation:

x-component

$$U_r\frac{\partial U_r}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta}-\frac{{U_\theta}^2}{r}+U_z\frac{\partial U_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\mu}{\rho}\left(\nabla^2 U_r-\frac{U_r}{r^2}-\frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta}\right) \quad \text{-------- (ii)}$$

y-component

$$U_r\frac{\partial U_\theta}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta}-\frac{U_r U_\theta}{r}+U_z\frac{\partial U_\theta}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial \theta}+\frac{\mu}{\rho}\left(\nabla^2 U_\theta-\frac{U_\theta}{r^2}+\frac{2}{r^2}\frac{\partial U_r}{\partial \theta}\right) \quad \text{-------- (iii)}$$

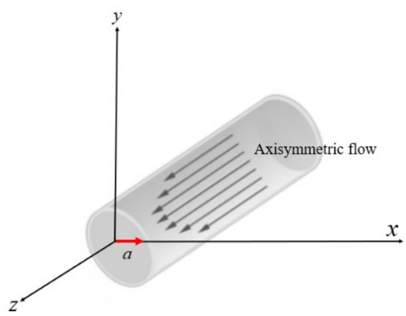
z-component

$$U_r\frac{\partial U_z}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_z}{\partial \theta}+U_z\frac{\partial U_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\nabla^2 U_z\right) \quad \text{-------- (iv)}$$

where $\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$

Case I: Steady flow in a pipe

- Considered a fully developed laminar motion through a tube of radius, a
- Flow through a tube is frequently called circular Poissule flow
- To solve the problem cylindrical coordinate $\left(r,\theta,z\right)$ need to be applied with the z axis coinciding with the axis of the pipe



- In this flow problem only velocity $U_z(r,\theta,z)$ is known zero components of velocity as the other $U_r = U_\theta = 0$

Continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(rU_r) + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} = 0$$

$$\frac{\partial U_z}{\partial z} = 0$$

$$U_z = U_z(r,\theta)$$

But axisymmetric flow (not depend on θ), then

$$U_z = U_z(r)$$

Momentum equation:

x-component

y-component

z-component

The equation of motion becomes:

$$\frac{\partial p}{\partial z} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right)\right) \text{ ------------ (v)}$$

And the boundary condition

U_z is bounded at $r = 0$
$U_z = 0$ at $r = a$

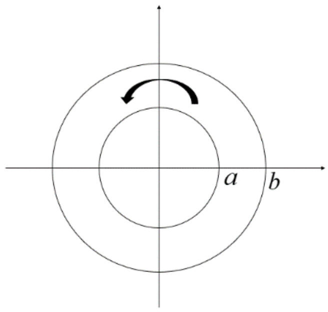
Solution of (v):

And boundary conditions:

U_z is bounded at $r = 0$

$U_z = 0$ at $r = a$

Case II: Steady flow between two rotating cylinders



- Consider the steady 2D velocity $\vec{U} = U_\theta \hat{\theta}$ in between concentric cylinder with radius $r = a$ and $r = b$ respectively and $a < b$
- In this concentric cylinder, the flow can only be steady if at least one of the cylinders is forces to rotate
- Therefore, the boundary condition can be stated as
  a) By considering continuity equation in cylindrical coordinate, show that

  $$U_\theta = U_\theta(r, z)$$

  b) By considering momentum equation in cylindrical coordinate, show that the equation of motion is

  $$-\frac{U_\theta^{\ 2}}{2} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

  $$0 = \frac{\mu}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_\theta}{\partial r}\right) - \frac{U_\theta}{r^2}\right]$$

  c) Show that by solving equation in

  $$U_\theta(r) = -\frac{\Omega a^2}{b^2 - a^2} r + \frac{\Omega a^2 b^2}{\left(b^2 - a^2\right) r}$$