CHAPTER 7
EULER'S EQUATION OF MOTION

7.1 DERIVATIVE OF EQUATION

- Apply the principal of linear momentum is a small 'dyed' blob of fluid of volume δv
- Allow for the present of a gravitational body force per unit mass, $\vec{F}$
- The total force on the blob is $\left(-\nabla p + \rho \vec{F}\right)\delta v$
- This force must be equal to the product of the blob mass, which is conserves and acceleration is given as $\rho \delta v \frac{D\vec{u}}{Dt}$
- Therefore, we have

$$\left(-\nabla p + \rho \vec{F}\right)\delta v = \rho \delta v \frac{D\vec{u}}{Dt}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{F} \quad \text{--------------} \ (1)$$

Equation (1) is known as momentum equation of Euler's equation. Together with the continuity equation $\nabla \cdot \vec{u} = 0$ ---------- (2)

both equations of motion for ideal fluid (inviscid)

7.2 SOLUTION OF EULER'S EQUATION

Consider the Euler's equation

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{F} \qquad \text{or}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \vec{F} \quad \text{-------------} \ (3)$$

We know that

$$\left(\vec{u}\cdot\nabla\right)\vec{u}=\nabla\left(\frac{1}{2}u^2\right)-\left(\vec{u}\times\vec{w}\right)$$

Substitute this identity into (3)

$$\frac{\partial\vec{u}}{\partial t}-\left(\vec{u}\times\vec{w}\right)=-\nabla\left(\frac{1}{2}u^2\right)-\frac{1}{\rho}\nabla p+\vec{F}\quad\text{------------- (4)}$$

If we set $\chi=\int\frac{dp}{\rho}$ hence, $\nabla\chi=\frac{1}{\rho}\nabla p$

From (4), we get

$$\frac{\partial\vec{u}}{\partial t}-\left(\vec{u}\times\vec{w}\right)=-\nabla\left(\frac{1}{2}u^2\right)-\nabla\chi+\vec{F}\quad\text{------------- (5)}$$

External force, $\vec{F}$ is conservative, a potential function ϕ such that

$$\vec{F}=-\nabla\phi$$

From (5), the equation becomes

$$\frac{\partial\vec{u}}{\partial t}-\left(\vec{u}\times\vec{w}\right)=-\nabla\left(\frac{1}{2}u^2+\chi+\phi\right)\quad\text{------------- (6)}$$

Equation (6) is known as the Euler's equation for the barotropic fluid

Case I: Steady and Irrotational Flow

From (6)

1) $\frac{\partial\vec{u}}{\partial t}=0$
2) $\vec{u}\times\vec{w}=0$

Therefore, (6) becomes

------------- (7)

Integrate (7)

------------- (8)

The fluid is incompressible, ρ is constant and the external force is a force due to gravity, $\vec{F}=-g\hat{k}$

Therefore, from (8)

------------- (9) (Bernoulli equation)

Case II: Steady and Rotational

From (6): $\frac{\partial \vec{u}}{\partial t} = 0$

But $\nabla\left(\frac{1}{2}u^2 + \chi + \phi\right)\cdot\frac{\vec{u}}{|\vec{u}|}$ means the directional derivative of function $\left(\frac{1}{2}u^2 + \chi + \phi\right)$ in the direction of $\frac{\vec{u}}{|\vec{u}|}$, so we have the derivation along the streamlines $\frac{1}{2}u^2 + \chi + \phi = c$, for any constant c

Case III: Unsteady and Irrotational Flow

From (6): $\vec{w} = 0 \;\Rightarrow\; \vec{u} = \nabla \phi$

7.3 PRESSURE GRADIENT

We consider the flow which is steady, irrotational incompressible fluid and influence by the gravity

Therefore,

1) $\frac{\partial \vec{u}}{\partial t}=0$

2) $\vec{w}=0 \Rightarrow \vec{u}=\nabla\phi$

3) $\nabla\cdot\vec{u}=0$

From the Bernoulli equation,

$$\frac{1}{2}u^2+gz+\frac{p}{\rho}=c$$

Suppose at $z=0 \Rightarrow p=p_0$ and $u=U$, where p_0 and U are constants, then

Therefore,

------------- (10)

a) If we consider a situation of no flow $(u=v=0)$, from (10)

b) If the flow moving without the influence of gravity, then (10) becomes

------------- (11)

There are two cases:

i. If $C_p = 0$

$\Rightarrow p = p_0$ means that the pressure is at free stream pressure

ii. If $C_p = 1$ (occurs when $\vec{u} = 0$)

$\Rightarrow p = p_0 + \frac{1}{2}\rho U^2$ means the pressure is at stagnation point

Here the stagnation point pressure is at greatest pressure that can be occur anywhere

Exercise

1) Find C_p for an irrotational flow over a fixed sphere with radius, a and the velocity potential, $\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$

2) Show that $\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$ is the solution for an irrotational flow about a fixed impermeable sphere, where the flow being uniform magnitude in the z-direction at infinity

3) A liquid of constant density, ρ is rotating about a vertical axis with angular velocity, $\Omega = \Omega\hat{k}$ as if it where a rigid body in a pressure of a uniform gravitational field, $\vec{F} = -g\hat{k}$ Compute the pressure distribution in this flow and show that the constant pressure surface is parabolic of concave upward.