SSCE 2393 NUMERICAL METHODS

CHAPTER 1

NONLINEAR EQUATIONS

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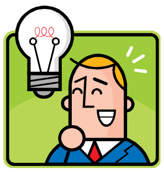
Chapter 1: Nonlinear Equations

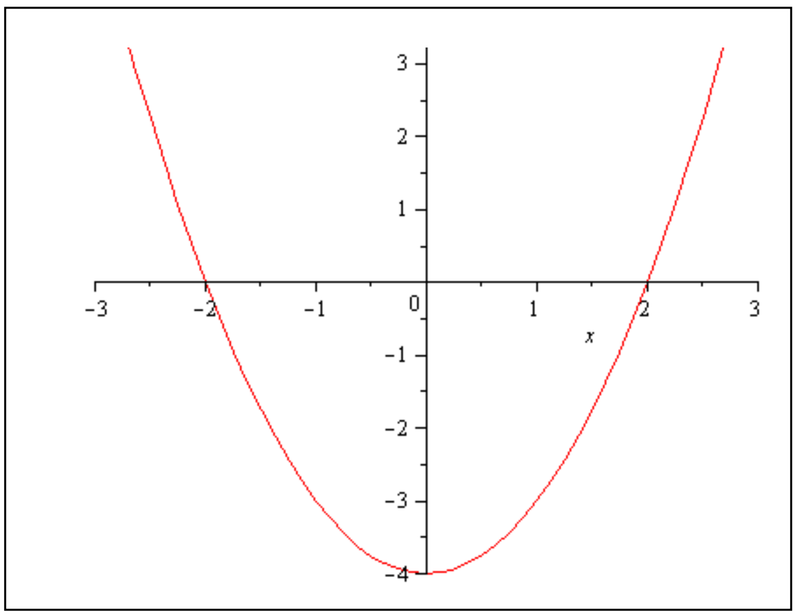
1.1 Introduction

In this chapter, we will learn how to find the root(s) of a non-linear equation.



Root/Solution/Zeroes: the value which, substituted in an equation, satisfies the equation.

Example 1: $y = x^2 - 4.$



From the graph, the roots of y are -2 and 2.

In real applications, it is not always easy to draw the graph to locate the roots. However, we can always find the interval [a, b] where the roots lie.

Steps:

1. Breaking the original equation into few equations.
2. Draw the graphs.
3. Identify the intersection points.
4. Determine the interval.

Example 2:

Find the interval [a, b] where the roots for $y = x^2 - x - 3$ lie.

Step 1: Breaking the original equation into few equations.

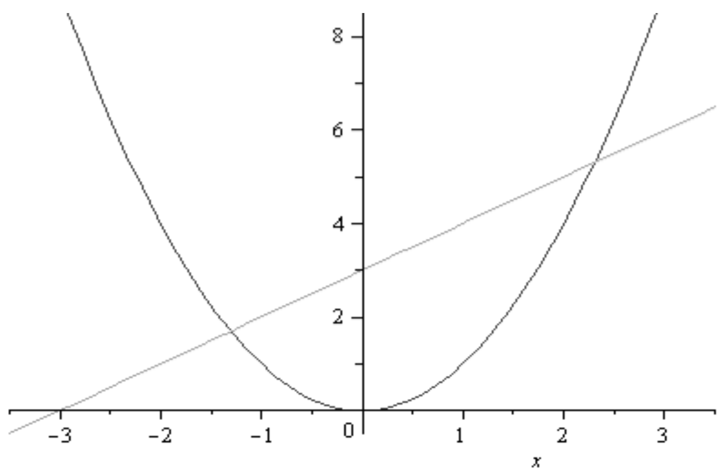
Set $y = 0$.

$$x^2 - x - 3 = 0$$

$$x^2 = x + 3$$

Set $y_1 = x^2$ and $y_2 = x + 3$

Step 2: Plot y_1 and y_2.



Step 3: Identify the intersection points.

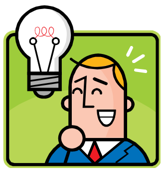
Step 4: Determine the intervals.

$$\therefore x^* \in [2,3] \text{ and } [-2,-1].$$

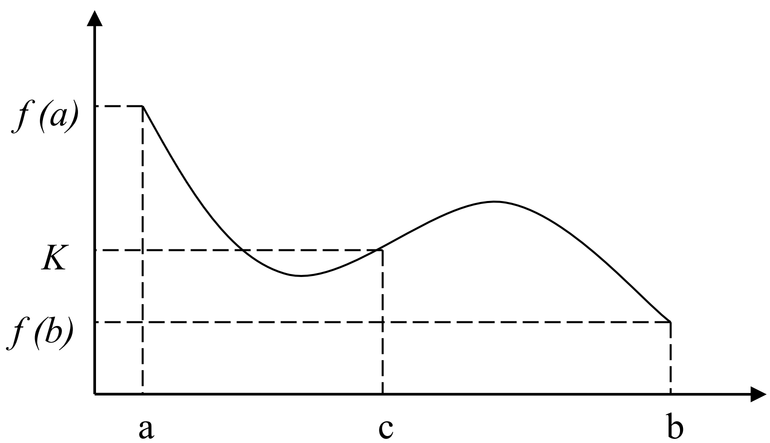
1.2 Intermediate Value Theorem (IVT)

- to verify the existence of the roots.

Theorem:

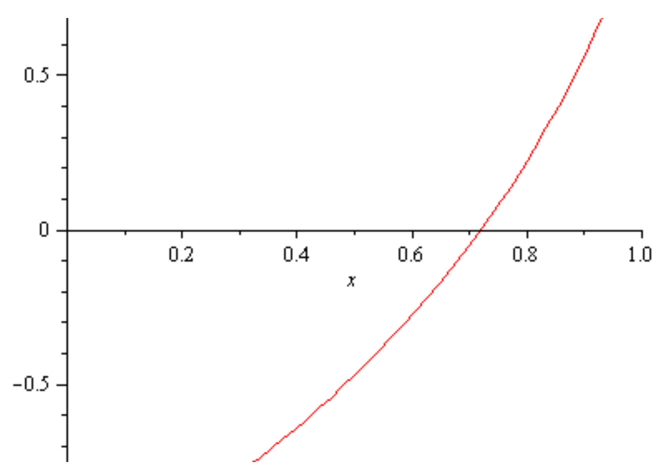


If $f(x) \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$ where $f(a)\,f(b)<0$, then there exist a number c in [a,b] for which $f(c) = K$.



Example 3:
Verify that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval [0,1].

Solution:



$$f(a) = f(0) = -1$$
$$f(b) = f(1) = 1$$
$$f(a)f(b) = -1 < 0$$
$$\therefore x^* \text{ exists in} [0,1]$$

1.3 BISECTION METHOD

- Formula : $c_i = \dfrac{a_i + b_i}{2}$
- Initial guess: a_0 and b_0 so that $f(a_0)f(b_0) < 0$
- Algorithm:

If $f(a_i)f(c_i) = 0$ then STOP. Take $x^* = c_i$ as a root.
If $f(a_i)f(c_i) < 0$ then $a_{i+1} = a_i$, $b_{i+1} = c_i$
If $f(a_i)f(c_i) > 0$ then $a_{i+1} = c_i$, $b_{i+1} = b_i$

- The process is repeated up to i-th step until it satisfies

  1. $f(c_i) = 0$ 2. $|f(c_i)| < \varepsilon$ or 3. $|b_i - a_i| < \varepsilon$

  for a given value of ε and then take $x^* \approx c_i$.

Example 4

Solve $f(x) = x^3 - x^2 - 2$ in the interval [1,2] take $\varepsilon = 0.005$. Stop when $|f(c_i)| < \varepsilon$.

Solution

$i = 0$

$a_0 = 1,\ b_0 = 2$ therefore $c_0 = \dfrac{1+2}{2} = 1.5$

$$\left.\begin{aligned} f(a_0) &= f(1) = 1^3 - 1^2 - 2 = -2 \\ f(c_0) &= f(1.5) = (1.5)^3 - (1.5)^2 - 2 = -0.875 \end{aligned}\right\} \Rightarrow f(a_0)f(c_0) > 0$$

Therefore: $a_1 = c_0 = 1.5$ and $b_1 = b_0 = 2$

$\underline{i=1}$

$a_1 = 1.5,\ b_1 = 2$ therefore $c_1 = \dfrac{1.5+2}{2} = 1.75$

$$\left.\begin{array}{l} f(a_1) = f(1.5) = -0.875 \\ f(c_1) = f(1.75) = 0.297 \end{array}\right\} \Rightarrow f(a_1)f(c_1) < 0$$

Therefore: $b_2 = c_1 = 1.75$ and $a_2 = a_1 = 1.5$

Notice that $|f(c_1)| > \varepsilon$,

so repeat the above step for $i = 2,3,..$until satisfying the conditions.

i	a_i	b_i	c_i	$f(a_i)$	$f(c_i)$	$\lvert f(c_i)\rvert$	$f(a_i)\,f(c_i)$
0	1	2	1.5	-2	-0.875	0.875	+
1	1.5	2	1.75	-0.875	0.297	0.297	-
2	1.5	1.75	1.625	-0.875	-0.350	0.350	+
6	1.688	1.704	**1.696**	-0.040	0.002	**0.002**	

From the table above, the algorithms stop at $i = 6$ since $|\ f(c_i)\ | < \varepsilon$.

Therefore, $\underline{x^* = c_6 = 1.696}$

1.4 FIXED-POINT ITERATION METHOD

- Formula: $x_{i+1} = g(x_i)$
- Convergence: $-1 \le g'(x_i) \le 1$ (or $|g'(x)| \le 1$)
- Initial guess: x_0 which is close to the root x^*.
- Do the iteration up to *i*-th step until $|x_i - x_{i-1}| < \varepsilon$ for a given value of ε and take $\underline{x^* \approx x_i}$.

Example 5

Find a root for $f(x) = x^3 - x^2 - 2$ in the interval [1, 2], and take $\varepsilon = 0.005$.

Solution

Step 1: find *g(x)*

Set *f(x) = 0* to get *x = g(x)*

$$\Rightarrow x^3 - x^2 - 2 = 0$$

$$x^3 = x^2 + 2$$

$$x = \left(x^2 + 2\right)^{\frac{1}{3}}$$

$$\therefore \quad g(x) = \left(x^2 + 2\right)^{\frac{1}{3}} \quad \text{and} \quad g'(x) = \frac{2x}{3}\left(x^2 + 2\right)^{\frac{-2}{3}}$$

$$\boxed{\therefore x_{i+1} = (x_i^2 + 2)^{\frac{1}{3}}}$$

Step 2: Iteration

<u>*i* = 0</u>

Let $x_0 = 1$

$$x_1 = (x_0^2 + 2)^{\frac{1}{3}}$$

$$= (1^2 + 2)^{\frac{1}{3}} = 1.442$$

and $|g'(x_0)| = \left| \frac{2(1)}{3}(1^2 + 2)^{\frac{-2}{3}} \right| = 0.320$ satisfies $-1 \le g'(x_i) \le 1$.

<u>*i* = 1</u>

$x_1 = 1.442$

$$x_2 = (x_1^2 + 2)^{\frac{1}{3}}$$

$$= ((1.442)^2 + 2)^{\frac{1}{3}} = 1.598$$

and $|g'(x_1)| = 0.377$ satisfies $-1 \le g'(x_i) \le 1$.

Error = $|x_1 - x_0| = |1.442 - 1| = 0.442 > \varepsilon$, so continue for *i* = 2

Proceed until the stopping condition is met.

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	1	
1	1.442	0.442
2	1.598	0.156
6	**1.693**	**0.004**

i = 6 to get $x^* = x_6 = 1.693$

Example 2:

Show that $x^2 - 3x + e^x - 2 = 0$ can be manipulated to form $x = \dfrac{x^2 + e^x - 2}{3}$. Then, find a root, x^* using fixed-point iteration method. Use $x_0 = -1$.

Solution:

$$x^2 - 3x + e^x - 2 = 0$$

$$3x = x^2 + e^x - 2$$

$$x = \frac{x^2 + e^x - 2}{3} \text{ (shown).}$$

$$x = g(x) = \frac{x^2 + e^x - 2}{3} \longrightarrow x_{i+1} = g(x_i) = \frac{x_i^{\,2} + e_i^{\,x} - 2}{3}$$

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	-1	
1	-0.2107	0.7893
2	-0.3819	0.1712
3		
4		

$$\therefore \quad \underline{x^* \approx x_4 = -0.3903}$$

1.5 NEWTON-RAPHSON METHOD

Formula : $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Initial guess: $x_0 = a$ or $x_0 = b$ so that $f(a_0)f(b_0) < 0$

Do the iteration up to *i*-th step until
$f(x_i) = 0$, $|f(x_i)| < \varepsilon$ or $|x_i - x_{i-1}| < \varepsilon$ for a given value of ε
and take $x^* \approx x_i$.

<u>Example 1</u>

Find a root for $f(x) = x^3 - x^2 - 2$ in the interval [1, 2] and take $\varepsilon = 0.005$. Stop when $|x_i - x_{i-1}| < \varepsilon$.

Solution

**you can choose any number between 1 and 2 to start,* x_0 *.*

$$f(x) = x^3 - x^2 - 2$$
$$f'(x) = 3x^2 - 2x$$

<u>$i = 0$</u>

$$x_0 = 1, \quad \text{so } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{-2}{1} = 3$$

Continue for next *i* = 1

$\underline{i=1}$

$$x_1 = 3, \quad \text{so } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3 - \frac{16}{21} = 2.238$$

Notice that error = $|x_1 - x_0| = |3-1| = 2 > \varepsilon$, so continue for $i = 2$.

i	x_i	$f(x_i)$	$f'(x_i)$	$\lvert x_i - x_{i-1}\rvert$
0	1	-2	1	
1	3	16	21	2
2	2.238	4.201	10.550	0.762
6	1.696	0.002	5.237	**0.000**

Stop the algorithm at $i = 6$ to get $x^* = x_6 = 1.696$ where $|x_6 - x_5| = 0 < \varepsilon$