SSCE 2393 NUMERICAL METHODS

CHAPTER 2

LINEAR EQUATION SYSTEMS

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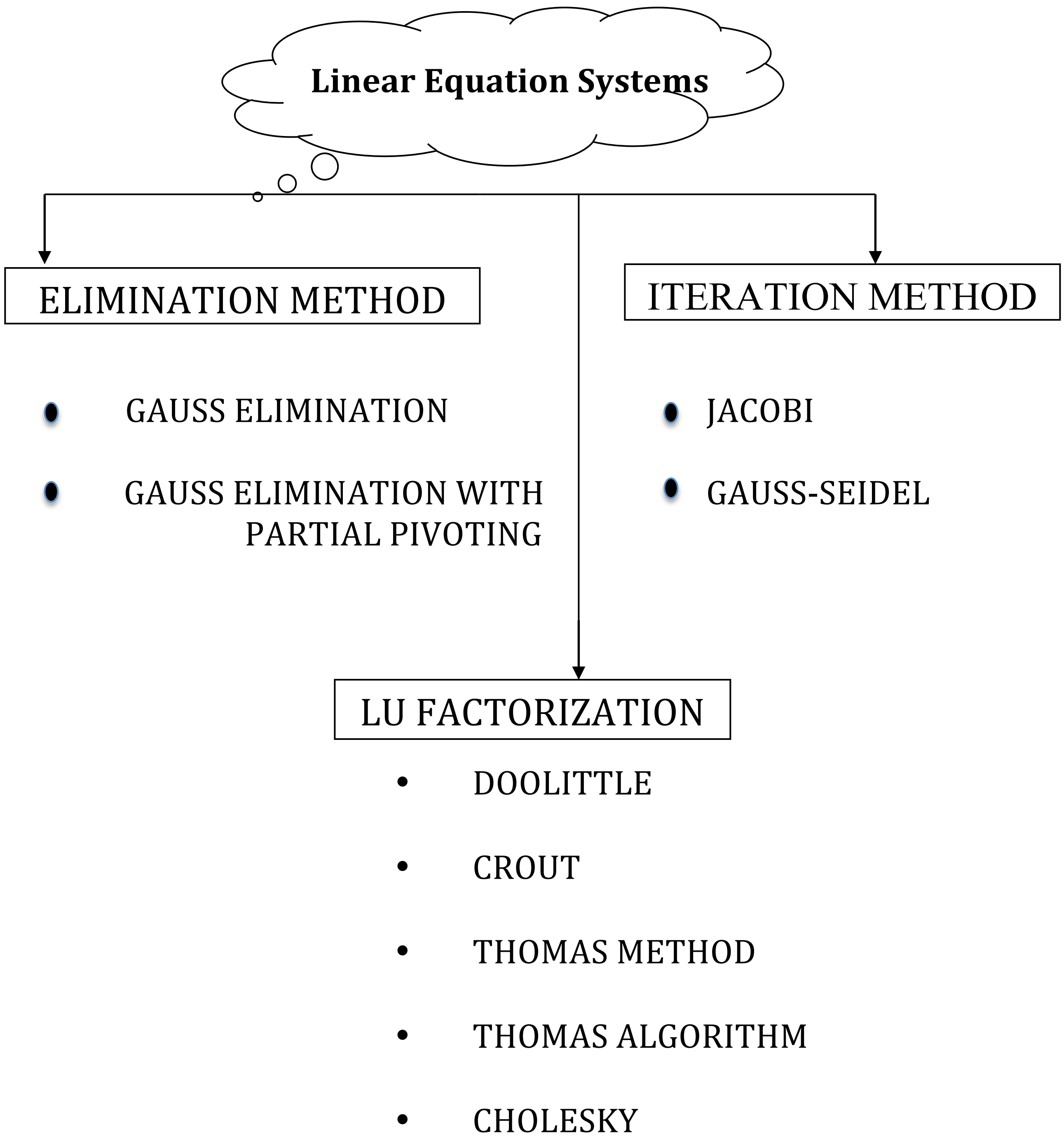
Overview Chapter 2



2.1 INTRODUCTION OF LINEAR EQUATION SYSTEMS

Linear equations: $\mathbf{ax} = \mathbf{b}$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \; &\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

In *matrix form* ($\mathbf{A}_{\mathrm{m\times n}}\mathbf{x} = \mathbf{b}$)

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In *augmented matrix* form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_n \end{array}\right]$$

We will only consider $\mathbf{A}_{\mathrm{n\times n}}$ matrix (square matrix).

Solutions →
- Elimination method
- LU
- Iteration

2.1 Elimination Method

- **Gauss Elimination**
- **Gauss elimination with partial pivoting**

2.1.1 Gauss Elimination

Given:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Written in augmented matrix form as:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \xrightarrow{\text{Elementary row operations}} \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & f_1 \\ 0 & u_{22} & u_{23} & f_2 \\ 0 & 0 & u_{33} & f_3 \end{array}\right]$$

$$\mathbf{Ax} = \mathbf{b} \qquad\qquad \mathbf{Ux} = \mathbf{f}$$

Elementary row operations ↓

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

← Backward subst.

$$x_3 = f_3 / u_{33}$$
$$x_2 = (f_2 - u_{33}x_3) / u_{22}$$
$$x_1 = (d_1 - u_{12}x_2 - u_{13}x_3) / u_{33}$$

Example 1:

Solve the following linear system using Gauss elimination method.

$$2.51x_1 + 1.48x_2 + 4.53x_3 = 5.56$$
$$1.48x_1 + 0.93x_2 - 1.30x_3 = -0.75$$
$$2.68x_1 + 3.04x_2 - 1.48x_3 = -1.84$$

Solution

$$\left[\begin{array}{ccc|c} 2.51 & 1.48 & 4.53 & 5.56 \\ 1.48 & 0.93 & -1.30 & -0.75 \\ 2.68 & 3.04 & -1.48 & -1.84 \end{array}\right] \xrightarrow{B_2 \to -0.59B_1 + B_2} \left[\begin{array}{ccc|c} 2.51 & 1.48 & 4.53 & 5.56 \\ 0 & 0.06 & -3.97 & -4.03 \\ 2.68 & 3.04 & -1.48 & -1.84 \end{array}\right]$$

multiplier, $m_{21} = a_{21}/a_{11} = 0.59$ multiplier, $m_{31} = a_{31}/a_{11} = 1.07$

$$\xrightarrow{B_3 \to -1.07B_1 + B_3} \left[\begin{array}{ccc|c} 2.51 & 1.48 & 4.53 & 5.56 \\ 0 & 0.06 & -3.97 & -4.03 \\ 0 & 1.46 & -6.33 & -7.79 \end{array}\right] \xrightarrow{B_3 \to -24.33B_2 + B_3}$$

multiplier, $m_{32} = a_{32}/a_{22} = 24.33$

$$\longrightarrow \left[\begin{array}{ccc|c} 2.51 & 1.48 & 4.53 & 5.56 \\ 0 & 0.06 & -3.97 & -4.03 \\ 0 & 0 & 90.26 & 90.26 \end{array}\right]$$

$\therefore$ Using backward substitution, we get:

$$x_3 = 1$$
$$x_2 = -1$$
$$x_1 = 1$$

Exercise:

Solve the following linear systems using Gauss elimination method:

1. Use 4 decimal places.

$$\begin{bmatrix} 5 & -1 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix}$$

2. Use 3 decimal places.

$$\begin{aligned} 8x_1 + 2x_2 - x_3 + 2x_4 &= -2 \\ x_1 - 8x_2 + x_3 - 2x_4 &= 4 \\ 2x_1 - x_2 + 7x_3 - x_4 &= -1 \\ -2x_1 + x_2 - 3x_3 - 8x_4 &= 3 \end{aligned}$$

Ans: $x_3 = -1.0555,\ x_2 = 1.7222,\ x_1 = 2.5555$

$x_4 = -0.325,\ x_3 = -0.231,\ x_2 = -0.458,\ x_1 = -0.083$

2.1.2 Gauss Elimination with Partial Pivoting

why????

- to solve problem that involve division by zero (in case $a_{11} = 0$ or $a_{22} = 0$)
- reduce the round-off errors.

The algorithm follows the Gauss elimination method **except:**

- Interchange rows when needed at the *k*-th step so that the absolute value of pivot element a_{kk} is the largest element compare to the other elements underneath the pivot.

Then, do the elimination process.

Example 2:

Use Gauss elimination method with partial pivoting to solve:

$$\begin{bmatrix} 0 & 0 & 3 & 4 \\ 2 & 9 & 1 & 0 \\ 0 & 1 & 9 & 4 \\ 5 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 8 \\ 9 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|c} 0 & 0 & 3 & 4 & 8 \\ 2 & 9 & 1 & 0 & 6 \\ 0 & 1 & 9 & 4 & 8 \\ 5 & 1 & 0 & 0 & 9 \end{array}\right) B_1 \leftrightarrow B_4 \left(\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 9 \\ 2 & 9 & 1 & 0 & 6 \\ 0 & 1 & 9 & 4 & 8 \\ 0 & 0 & 3 & 4 & 8 \end{array}\right) \xrightarrow[B_2 := -m_{21}B_1 + B_2]{m_{21} = \frac{2}{5}}$$

$$\left(\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 9 \\ 0 & 8.6 & 1 & 0 & 2.4 \\ 0 & 1 & 9 & 4 & 8 \\ 0 & 0 & 3 & 4 & 8 \end{array}\right) \xrightarrow[B_3 := -m_{32}B_2 + B_3]{m_{32} = \frac{1}{8.6}} \left(\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 9 \\ 0 & 8.6 & 1 & 0 & 2.4 \\ 0 & 0 & 8.8837 & 4 & 7.7209 \\ 0 & 0 & 3 & 4 & 8 \end{array}\right)$$

$$\xrightarrow[B_4 := -m_{43}B_3 + B_4]{m_{43} = \frac{3}{8.8837}} \left(\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 9 \\ 0 & 8.6 & 1 & 0 & 2.4 \\ 0 & 0 & 8.8837 & 4 & 7.7209 \\ 0 & 0 & 0 & 2.6492 & 5.3927 \end{array}\right)$$

Then we have:

$$\begin{aligned} 5x_1 + x_2 &= 9 \\ 8.6x_2 + x_3 &= 2.4 \\ 8.8837x_3 + 4x_4 &= 7.7209 \\ 2.6492x_4 &= 5.3927 \end{aligned}$$

Use backward subst. to get

$$\begin{aligned} x_4 &= 2.0356 \\ x_3 &= -0.0474 \quad \text{or} \quad \mathbf{x} = \hat{\mathbf{x}} = (1.743,\ 0.285,\ -0.048,\ 2.036) \\ x_2 &= 0.2846 \\ x_1 &= 1.7431 \end{aligned}$$

Exercise:

Solve the following systems using Gauss elimination method with partial pivoting. Use 3 decimal places.

1.

$$
\begin{aligned}
2x_1 - x_2 + 7x_3 - x_4 &= -1 \\
-2x_1 + x_2 - 3x_3 - 8x_4 &= 3 \\
8x_1 + 2x_2 - x_3 + 2x_4 &= -2 \\
x_1 - 8x_2 + x_3 - 2x_4 &= 4
\end{aligned}
$$

2.

$$
\begin{pmatrix} 4 & 1 & -1 \\ 5 & 1 & 2 \\ 6 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}
$$

Ans: 1. $x_4 = -0.325$ $x_3 = -0.231$ $x_2 = -0.458$ $x_1 = -0.083$

2. $x_1 = 3$ $x_2 = -13$ $x_3 = 1$.

2.2 LU FACTORIZATION METHOD

For a linear system

$$Ax = b$$

Use substitution of $A = LU$, where L is a **lower triangular matrix**, and U is **upper triangular matrix**.

$$LUx = b$$

Let

$$Ux = Y$$

Yields

$$LY = b$$

$$L = \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix} \qquad U = \begin{bmatrix} k & l & m & n \\ 0 & u & p & q \\ 0 & 0 & r & s \\ 0 & 0 & 0 & t \end{bmatrix}$$

Procedure:

Step 1: From $A = LU$, solve for L and U.

Step 2: From $LY = b$, solve for Y by forward substitution.

Step 3: From $Ux = Y$, solve for x by backward substitution.

2.2.1 Doolittle Method

$$A = LU$$

Diagonal element for matrix $L = 1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ d & e & 1 & 0 \\ g & h & i & 1 \end{bmatrix}$

$$Ax = b$$

objective→ to get the value of x

Step:

1. $A = LU$, find the matrix for L and U

2. $Ly = b$, solve for y

3. $Ux = y$, solve for x

(Steps 2 and 3:) use backward and forward substitution

Example 3:

Solve this equation system using Doolittle method.

$$\begin{aligned} 3x_1 + 2x_2 + 9x_3 &= 28 \\ 2x_1 - \ \ x_2 + 6x_3 &= 14 \\ 5x_1 + 2x_2 - 4x_3 &= -13 \end{aligned}$$

Do the calculation in 4 decimal places.

Example 4:

Solve this equation system using Doolittle method.

$$\begin{aligned} &2x_1 = 3 \\ &x_1 + 1.5x_2 = 4.5 \\ &-3x_2 + 0.5x_3 = -6.6 \\ &x_4 + x_3 + 2x_1 - 2x_2 - 0.8 = 0 \end{aligned}$$

2.2.2 Crout Method

$$A = LU$$

Diagonal element for matrix $U = 1$

$$\begin{bmatrix} 1 & l & m & n \\ 0 & 1 & p & q \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ax = b$$

objective→to find the value of x

Step:

1. $A = LU$, determine L and U
2. $Ly = b$, solve for y
3. $Ux = y$, solve for x

(steps 2 and 3) use forward and backward substitutions

Example 5:

Solve this linear system using Crout method.

$$\begin{aligned} 3x_1 + 2x_2 + 9x_3 &= 28 \\ 2x_1 - x_2 + 6x_3 &= 14 \\ 5x_1 + 2x_2 - 4x_3 &= -13 \end{aligned}$$

2.2.3 Thomas Method

$$Ax = b$$

Check this first!

Make sure that matrix A must
be **Tridiagonal Matrix**

$$\begin{bmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & 0 & 0 & c_n & d_n \end{bmatrix}$$

If not → **rearrange the matrix**

$$L = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ c_2 & \alpha_2 & 0 & 0 & 0 \\ 0 & c_3 & \alpha_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & c_n & \alpha_n \end{bmatrix} \qquad U = \begin{bmatrix} 1 & \beta_1 & 0 & 0 & 0 \\ 0 & 1 & \beta_2 & 0 & 0 \\ 0 & 0 & 1 & \beta_3 & 0 \\ 0 & 0 & \ddots & \ddots & \beta_{n-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ax = b$$

objective→ to find the value of x

Step:

1. $A = LU$, determine the matrix for L and U

2. $Lw = b$, solve for y } use forward and backward substitutions

3. $Ux = w$, solve for x

Example 6:

Solve this linear system equation using Thomas method.

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 4 & -1 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -20 \\ 11 \end{bmatrix}$$

2.2.4 Thomas Algorithm

Generalization of Thomas method

- suitable to solve large system
- Easy for programming coding

Remember! Always check whether Matrix A is tridiagonal matrix or not…

Check this first!

$$\begin{bmatrix} d_1 & e_1 & 0 & 0 & 0 \\ c_2 & d_2 & e_2 & 0 & 0 \\ 0 & c_3 & d_3 & e_3 & 0 \\ 0 & 0 & \ddots & \ddots & e_{n-1} \\ 0 & 0 & 0 & c_n & d_n \end{bmatrix}$$

If not → **rearrange the matrix**

$$\begin{bmatrix} d_1 & e_1 & 0 & 0 & 0 \\ c_2 & d_2 & e_2 & 0 & 0 \\ 0 & c_3 & d_3 & e_3 & 0 \\ 0 & 0 & \ddots & \ddots & e_{n-1} \\ 0 & 0 & 0 & c_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Formula:

Thomas's Algorithm:

$$\alpha_1 = d_1$$
$$\alpha_i = d_i - c_i\beta_{i-1}\ , \quad i = 2, 3, ..., n$$
$$\beta_i = \frac{e_i}{\alpha_i}\ , \ i = 1, 2, 3, ..., n-1$$
$$y_1 = b_1 / \alpha_1$$
$$y_i = (b_i - c_i y_{i-1}) / \alpha_i\ , \quad i = 2, 3, ..., n$$
$$x_n = y_n$$
$$x_i = y_i - \beta_i x_{i+1}\ , \quad i = n-1, n-2, ..., 1$$

Table:

i	1	2	…	n
d_i				
e_i				
c_i				
b_i				
α_i				
β_i				
y_i				
x_i				

Example 7:

Solve this linear system using Thomas method.

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 4 & -1 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -20 \\ 11 \end{bmatrix}$$

Example 8:

Given

$$3x_3 + 4x_4 = 8$$
$$2x_1 + 9x_2 + x_3 = 6$$
$$x_2 + 9x_3 + 4x_4 = 8$$
$$2x_1 + x_2 = 9$$

(i) What condition that needs to be fulfilled before using Thomas's Algorithm?

(ii) If the system above satisfied the condition, solve the equation system.

2.2.5 Cholesky Method

$$Ax = b$$

Matrix A must be **symmetric positive-definite**

<u>Definition</u>

$$x^T Ax > 0, \quad \forall x \neq 0$$

<u>Rules (theorem)</u>

1. $|A| \neq 0$
2. $a_{ii} > 0, \forall i = 1,2,\dots,n$
3. $\max\limits_{\substack{1 \le k \le n \\ 1 \le j \le n}} |a_{kj}| \le \max\limits_{1 \le i \le n} |a_{ii}|$
4. $(a_{ij})^2 < a_{ii}a_{jj}, \ \forall_{i \neq j} \ i, j = 1,2,\dots,n$

$$A = LU$$

where $U = L^T$

$$Ax = b$$

Target→ to find the value of x

Step:

1. $A = LU$, determine L and U
2. $Ly = b$, solve for y
3. $Ux = y$, solve for x

(Steps 2 and 3: use forward and backward substitutions)

Example 9:

Show that matrix *A* for the following linear system is symmetric positive-definite by using definition. Then, solve the system of linear equations by Cholesky Method.

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 7 \\ -x_1 + 7x_2 + 3x_3 &= 6 \\ x_1 + 3x_2 + 5x_3 &= 1 \end{aligned}$$

2.3 ITERATIVE METHOD

Must satisfy convergence criterion i.e.
A must be strictly diagonally dominant matrix

Is A Strictly Diagonally Dominant Matrix?

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

if not → **rearrange rows.**

e.g.

Show that $\begin{bmatrix} 3 & 2 & 0 \\ 2 & -5 & 1 \\ 1 & 2 & 7 \end{bmatrix}$ is a SDD matrix.

Solution:

$B1: \; |3| > |2| + |0|$

$B2: |-5| > |2| + |1|$

$B3: |7| > |2| + |1|$

2.3.1 Jacobi Method

Formula:

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}}{a_{33}}$$

Initial guess: $x^{(0)} = (0 \quad 0 \quad \ldots \quad 0)^T$

Stop the iteration when:

$$\max_{1 \le i \le n} \left\{ \left| x_i^{(k)} - x_i^{(k-1)} \right| \right\} < \varepsilon$$

for a given value of ε and take $\mathbf{x} \approx \mathbf{x}^{(k)}$

Example 10:

Solve the following linear system using Jacobi method.
Set $\mathbf{x}^{(0)} = \mathbf{0}$. Take $\varepsilon = 0.05$.

$$\begin{aligned} x_1 - 3x_2 + 12x_3 &= 31 \\ 4x_1 + x_2 - x_3 &= 3 \\ 2x_1 + 7x_2 + x_3 &= 19 \end{aligned}$$

Solution

Is matrix ***A*** SDD?

Rearrange rows :

$$4x_1 + x_2 - x_3 = 3$$
$$2x_1 + 7x_2 + x_3 = 19$$
$$x_1 - 3x_2 + 12x_3 = 31$$

The iteration formula:

$$x_1^{(k+1)} = \frac{3 - x_2^{(k)} + x_3^{(k)}}{4}$$
$$x_2^{(k+1)} = \frac{19 - 2x_1^{(k)} - x_3^{(k)}}{7}$$
$$x_3^{(k+1)} = \frac{31 - x_1^{(k)} + x_2^{(k)}}{12}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|x_1^{(k)} - x_1^{(k-1)}\right\|$	$\left\|x_2^{(k)} - x_2^{(k-1)}\right\|$	$\left\|x_3^{(k)} - x_3^{(k-1)}\right\|$
0	0	0	0			
1						
2						
3						
4						
5						

$$\left\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\right\| = \max_{1 \le i \le 3} \left\{ \left|x_i^{(5)} - x_i^{(4)}\right| \right\}$$
$$= \max \{0.01,\ 0.01,\ 0.01\} = 0.01 < \varepsilon$$

$$\therefore \mathbf{x} \approx \mathbf{x}^{(5)} = (1.01,\ 2.00,\ 3.00)$$

2.3.2 Gauss Seidel Method

Formula:

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}}{a_{22}}$$

$$x_3^{(k+1)} = \frac{b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}}{a_{33}}$$

Initial guess: $x^{(0)} = (0 \quad 0 \quad \ldots \quad 0)^T$

Stop the iteration when:

$$\max_{1 \le i \le n} \left\{ \left| x_i^{(k+1)} - x_i^{(k)} \right| \right\} < \varepsilon$$

for a given value of ε and take $\mathbf{X} \approx \mathbf{X}^{(k)}$

Example 11:

Solve the following linear system using Gauss-Seidel method.
Set $\mathbf{x}^{(0)} = \mathbf{0}$. Take $\varepsilon = 0.05$.

$$x_1 - 3x_2 + 12x_3 = 31$$
$$4x_1 + x_2 - x_3 = 3$$
$$2x_1 + 7x_2 + x_3 = 19$$

Solution

Is matrix ***A*** SDD?

Rearrange rows :

$$4x_1 + x_2 - x_3 = 3$$
$$2x_1 + 7x_2 + x_3 = 19$$
$$x_1 - 3x_2 + 12x_3 = 31$$

The iteration formula:

$$x_1^{(k+1)} = \frac{3 - x_2^{(k)} + x_3^{(k)}}{4}$$

$$x_2^{(k+1)} = \frac{19 - 2x_1^{(k+1)} - x_3^{(k)}}{7}$$

$$x_3^{(k+1)} = \frac{31 - x_1^{(k+1)} + x_2^{(k+1)}}{12}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|x_1^{(k)} - x_1^{(k-1)}\right\|$	$\left\|x_2^{(k)} - x_2^{(k-1)}\right\|$	$\left\|x_3^{(k)} - x_3^{(k-1)}\right\|$
0	0	0	0			
1						
2						
3						
4						

$$\left\|\mathbf{x}^{(4)} - \mathbf{x}^{(3)}\right\| = \max_{1 \le i \le 3} \left\{ \left| x_i^{(4)} - x_i^{(3)} \right| \right\}$$
$$= \max \{0.00,\ 0.00,\ 0.00\} = 0.00 < \varepsilon$$

$$\therefore\ \mathbf{x} \approx \mathbf{x}^{(5)} = (1.00,\ 2.00,\ 3.00)$$

Exercise:

1. Write the Gauss - Seidel formula for

$$\begin{aligned} 5x_1 - x_2 + x_3 &= 11 \\ 2x_1 + 8x_2 - x_3 &= 17 \\ -x_1 + x_2 + 7x_3 &= 21 \end{aligned}$$

Then find the value of x_1, x_2, and x_3. Do the calculation in 3 decimal places.

2. Solve the following system using Gauss – Seidel method and stop the iteration when $\left\| \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\|_\infty < 0.0005$.

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$$