SSCE 2393 NUMERICAL METHODS

CHAPTER 3

INTERPOLATION

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Overview of Chapter 3

3.1 What is interpolation

3.2 Lagrange Interpolation

3.3 Newton's divided difference

3.4 Newton's forward difference

3.5 Newton's backward difference

3.1 What is Interpolation

- Is the process of estimating an intermediate value of a set data.

- Suppose $\{(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)\}$ be the set of $(n+1)$ given points. The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n, is called interpolation.

- One of simplest approach to interpolation is based on polynomial.

- Idea behind interpolation is to find a polynomial which agrees with specified data points.

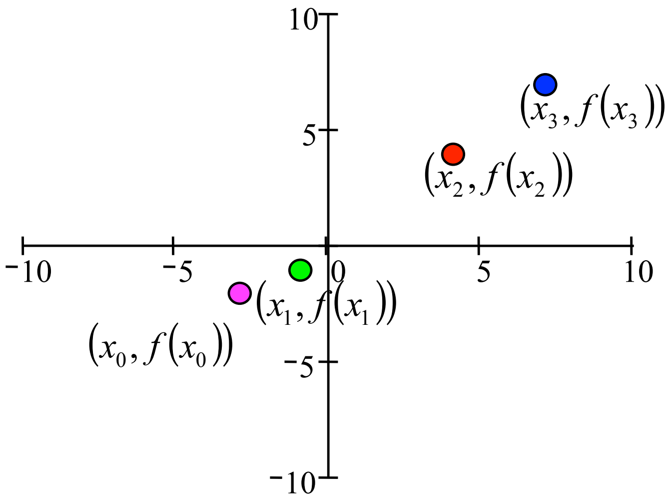
- We seek a polynomial interpolation $P_n(x)$ of degree $\leq n$ such that:

$$P_n(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n.$$

- This polynomial can then be used to generate approximate values at other points between x_0 and x_n

Aim: To find a function $P_n(x)$ that exactly represents a collection of data.

(a)



(b)

n	0	1	..
x_n	..	..	..
$f(x_n)$	..	..	..

(c) $f(x_0) = a, f(x_1) = b, ..$

3.2 Lagrange Polynomial Interpolation

General formula for Lagrange polynomial Interpolation which passing through all the points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)),$$

is given as:

$$P_n(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + .. + L_n(x) f(x_n)$$

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$$P_n(x) = L_0(x) y_0 + L_1(x) y_1 + .. + L_n(x) y_n$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\left(x - x_j\right)}{\left(x_i - x_j\right)}$$

is the Lagrange coefficient that satisfies

$$\sum_{i=0}^{n} L_i(x) = 1$$

Example 1

Find the Lagrange polynomial interpolating for data $f(0)=1,\ f(2)=-1,\ f(4)=-1$ and $f(6)=1$. Hence evaluate $f(3)$ and $f(5)$.

Example 2

a) Estimate log4 between log3 and log5 by using linear Lagrange interpolation. Then compare the result with the exact solution.

b) The density of a chemical material at three different temperatures is given below:

Temperature (^0C)	90	200	300
Density (kg/cm^3)	950	900	850

i) Write the suitable Lagrange formula for the above data.

ii) Estimate the density of above chemical material when temperature is $250\,^0C$.

3.3 Newton's divided - difference formula

General Formula for Newton's divided- difference that passing through all the points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)),$$

is given as:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

The coefficient $a_0, a_1, .., a_n,$ → Newton's divided-difference.

The value of divided-difference for

$$\{(x_i, f_i)\}_{i=0}^{n}$$

is defined as:

$$f_i^{[0]} = f_i$$

$$f_i^{[j]} = \frac{f_{i+1}^{[j-1]} - f_i^{[j-1]}}{x_{i+j} - x_i}, \quad j = 1, 2, .., n-1$$

($f_i^{[j]}$ is define as j-th divided - difference), and given in the table below:

Table for Newton's divided –difference

I	x_i	$f_i[0]$	$f_i[1]$	$f_i[2]$	$\cdots$	$f_i[n-1]$	$f_i[n]$
0	x_0	f_0	$f_0^{[1]}$	$f_0^{[2]}$		$f_0^{[n-1]}$	$f_0^{[n]}$
1	x_1	f_1	$f_1^{[1]}$	$f_1^{[2]}$		$f_1^{[n-1]}$	
$\vdots$	$\vdots$						
$n-2$	x_{n-2}	f_{n-2}	$f_{n-2}^{[1]}$	$f_{n-2}^{[2]}$			
$n-1$	x_{n-1}	f_{n-1}	$f_{n-1}^{[1]}$				
n	x_n	f_n					

Given (columns x_i, $f_i[0]$); From Newton's divided -difference formula (remaining columns)

We also can write;

$$a_0 = f_0 = f_0^{[0]}$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = \frac{f_1^{[0]} - f_0^{[0]}}{x_1 - x_0} = f_0^{[1]}$$

$$a_2 = \frac{\dfrac{f_2 - f_1}{x_2 - x_1} - \dfrac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = \frac{\dfrac{f_2^{[0]} - f_1^{[0]}}{x_2 - x_1} - \dfrac{f_1^{[0]} - f_0^{[0]}}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{f_1^{[1]} - f_0^{[1]}}{x_2 - x_0} = f_0^{[2]}$$

Therefore, the Newton's divided-difference polynomial interpolation for $\{(x_i, f_i)\}_{i=0}^{2}$ (quadratic polynomial)

can be written as:

$$P_2(x) = f_0^{[0]} + f_0^{[1]}(x - x_0) + f_0^{[2]}(x - x_0)(x - x_1)$$

Generally, **Newton's divided- difference formula,** with degree $\leq n$ is given by;

$$P_n(x) = f_0^{[0]} + f_0^{[1]}(x - x_0) + f_0^{[2]}(x - x_0)(x - x_1) + \ldots$$
$$f_0^{[n]}(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

or

$$P_n(x) = \sum_{j=0}^{n} f_0^{[j]}\left(\prod_{k=0}^{j-1}(x - x_k)\right)$$

Example 1

Given that $e^0 = 1,\ e^{0.5} = 1.6487,\ e^1 = 2.7183$, use Newton's Divided difference formula to estimate the value of $e^{0.25}$. Then, compute the absolute error.

Example 2

For a function f, the Divided Difference table is given by:

$x_0 = 0$	$f[x_0] = ?$		
		$f[x_0, x_1] = ?$	
$x_1 = 0.4$	$f[x_1] = ?$		$f[x_0, x_1, x_2] = \frac{50}{7}$
		$f[x_1, x_2] = 10$	
$x_2 = 0.7$	$f[x_2] = 6$		

Determine the missing entries, then approximate $f(0.3)$.

3.4 Interpolation for uniform data

Uniform data means the step size, *h* for x_i are all same ie;

$$(x_1 - x_0) = (x_2 - x_1) = .. = (x_n - x_{n-1}) = h$$

There several methods to treat uniform data such as;

a) Newton's forward- difference and
b) Newton's backward- difference

3.4.1 Newton's forward difference

Newton's forward- difference formula

$$P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \ldots + \frac{r(r-1)(r-2)\ldots(r-n)}{n!}\Delta^n f_0$$

$$\text{where } r = \frac{x - x_0}{h}$$

Table for Newton's forward -difference

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\cdots$	$\Delta^{n-1} f_i$	$\Delta^n f_i$
0	x_0	f_0	Δf_0	$\Delta^2 f_0$		$\Delta^{n-1} f_0$	$\Delta^n f_0$
1	x_1	f_1	Δf_1	$\Delta^2 f_1$		$\Delta^{n-1} f_1$	
$\vdots$	$\vdots$						
$n-2$	x_{n-2}	f_{n-2}	Δf_{n-2}	$\Delta^2 f_{n-2}$			
$n-1$	x_{n-1}	f_{n-1}	Δf_{n-1}				
n	x_n	f_n					

Given (x_i, f_i)

from Newton's forward-difference formula (Δf_i … $\Delta^n f_i$)

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$

Example 1

By using Newton's Forward Difference formula, show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
f (x)	1	4	11	16	13	-4

Example 2

Newton forward difference formula order three can be written as $f(x) = f(0) + Ax + Bx^2 + Cx^3$. Find the coefficients A, B and C if you are given the following data:
$f(0) = 3, f(1) = 2, f(2) = 7, f(3) = 24, f(4) = 59$ and $f(5) = 118$.
Then estimate the value of $f(0.1)$ by using the above formula. Can you get the Newton forward difference's formula for the fourth order from the given data? Explain your answer.

3.4.2 Newton's backward- difference

Newton's backward- difference formula

$$P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_n + \ldots + \frac{r(r+1)(r+2)\ldots(r+n)}{n!}\nabla^n f_n$$

$$\text{where } r = \frac{x - x_n}{h}$$

Table for Newton's backward- difference

i	x_i	f_i	∇f_i	$\nabla^2 f_i$	$\cdots$	$\nabla^{n-1} f_i$	$\nabla^n f_i$
0	x_0	f_0					
1	x_1	f_1	∇f_1				
2	x_2	f_2	∇f_2	$\nabla^2 f_2$			
$\vdots$	$\vdots$						
$n-1$	x_{n-1}	f_{n-1}	∇f_{n-1}	$\nabla^2 f_{n-1}$		$\nabla^{n-1} f_{n-1}$	
n	x_n	f_n	∇f_n	$\nabla^2 f_n$		$\nabla^{n-1} f_n$	$\nabla^n f_n$

Given (x_i, f_i); from Newton's backward- difference formula (∇f_i … $\nabla^n f_i$)

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1}$$

Example 1

Given

x	1.0	1.2	1.4	1.6	1.8
$f(x)$	0.0000	0.1823	0.3365	0.4700	0.5878

Find an approximation of $f(1.7)$ using the Newton's backward-difference formula. Do the calculation using 4 decimal places.

Example 2

Given the following data;

x	1	1.2	1.4	1.6	1.8	2.0
$f(x)$	0.6570	0.9039	0.9985	0.9407	0.7344	0.4121

By choosing the suitable data, estimate the value of $f(1.5)$ using **cubic interpolation's formula** of:

a) Newton forward difference,

b) Newton backward difference