SSCE 2393 NUMERICAL METHODS

CHAPTER 5

NUMERICAL DIFFERENTIATION

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Overview of Chapter 5

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a) Newton's divided

b) Newton's forward

5.0 Introduction to Numerical Differentiation

- The problem arises: in practical applications a function is only known at a few points.
- For example, we may measure the position of a car every minute via a GPS (Global Positioning System) unit, and we want to compute its speed. If the position is known as a continuous function of time, we can find the speed by differentiating this function. But, when the position is only known at isolated times, this is not possible.
- Thus, numerical approach is used.

5.1 Numerical Differentiation: List of formulae

Aim: To Find $f'(x)$ and $f''(x)$ using numerical approach.

			Formula
f'	2-points	Forward	$f'(x)=\dfrac{f(x+h)-f(x)}{h}$
		Backward	$f'(x)=\dfrac{f(x)-f(x-h)}{h}$
		Centered	NIL
	3-points	Forward	$f'(x)=\dfrac{-f(x+2h)+4f(x+h)-3f(x)}{2h}$
		Backward	$f'(x)=\dfrac{3f(x)-4f(x-h)+f(x-2h)}{2h}$
		Centered	$f'(x)=\dfrac{f(x+h)-f(x-h)}{2h}$
f''	3-points	Forward	$f''(x)=\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2}$
		Backward	$f''(x)=\dfrac{f(x)-2f(x-h)+f(x-2h)}{h^2}$
		Centered	$f''(x)=\dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$
		Backward	$f^{(4)}(x)=\dfrac{f(x)-4f(x-h)+6f(x-2h)-4f(x-3h)+f(x-4h)}{h^4}$
		Centered	$f^{(4)}(x)=\dfrac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4}$

h = step size

How do they derive?

All of the above formulae are derived from the Taylor series.

General equations of Taylor series for $f(x+h)$:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \ldots + \frac{h^n f^{(n)}(x)}{n!} + \ldots$$

5.2 Deriving 1st derivative, $f'(x)$ using Taylor series. (ALL)

a) 2 points forward difference

$$f(x+h) \approx f(x) + hf'(x) \qquad (1a)$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad (1b)$$

b) 2 points backward difference

$$f(x-h) \approx f(x) - hf'(x) \qquad (2a)$$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \qquad (2b)$$

c) 3 points centered difference

(1a) – (2a) yields:

$$f(x+h) - f(x-h) \approx 2hf'(x) \qquad (3a)$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \qquad (3b)$$

d) 3 points forward difference

$$f(x+h) \approx f(x) + hf'(x) \qquad (4a)$$

$$f(x+2h) \approx f(x) + 2hf'(x) \qquad (4b)$$

4 x(4a) – (4b):

$$4f(x+h) \approx 4f(x) + 4hf'(x)$$

$$f(x+2h) \approx f(x) + 2hf'(x)$$

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x)$$

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h}$$

e) 3 points backward difference

$$f(x-h) \approx f(x) - hf'(x) \qquad (5a)$$
$$f(x-2h) \approx f(x) - 2hf'(x) \qquad (5b)$$

4 x(5a) – (5b):

$$4f(x-h) \approx 4f(x) - 4hf'(x)$$
$$f(x-2h) \approx f(x) - 2hf'(x)$$
$$4f(x-h) - f(x-2h) = 3f(x) - 2hf'(x)$$
$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}$$

5.3 Deriving 2nd derivative, $f''(x)$ using Taylor series. (Centered)

a) 3 points centered difference

$$f(x+h) \approx f(x) + hf'(x) + \frac{h^2}{2!} f''(x) \qquad (1a)$$

$$f(x-h) \approx f(x) - hf'(x) + \frac{h^2}{2!} f''(x) \qquad (1b)$$

(1a) + (1b)

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x)$$

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

Example 1:

Given the data below:

x	1.2	1.4	1.6	1.8
$f(x)$	2.5	3.6	3.8	3.5

a) Estimate $f'(1.6)$ using 3 points backward formula.

b) Estimate $f''(1.6)$ using 3 points centered formula.

Example 2:

Construct a table of value for

$$f(x) = \frac{\sin(x^3)\cos(2x)}{1+x}$$

when $x = 0.8, 1.0, 1.2, 1.4, 1.6.$ By using centered difference formulae, determine first derivative and second derivative for the above function at $x = 1.2$.

5.4 Finding 1st order and 2nd order derivatives using Newton's difference formula.

Recall from chapter 3, Newton's divided difference:

$$f(x) \approx P_n(x) = f_0^{[0]} + f_0^{[1]}(x - x_0) + f_0^{[2]}(x - x_0)(x - x_1) + \ldots$$

Therefore,

$$f'(x) = \frac{dP}{dx}$$

$$f''(x) = \frac{d}{dx}\left(\frac{dP}{dx}\right)$$

Also recall, Newton's forward difference:

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \ldots$$

where $r = \dfrac{x - x_0}{h}$.

Therefore,

$$f'(x) \equiv \frac{df}{dx} = \frac{dP}{dr} \cdot \frac{dr}{dx}$$

$$f''(x) = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(\frac{dP}{dr} \cdot \frac{dr}{dx}\right)$$

Example 1:

Given the table below:

x	1.0	1.4	1.8	2.2	2.6	3.0
$f(x)$	0.8415	0.9854	0.9738	0.8086	0.5105	0.1411

Estimate the value of $f'(2.0)$ and $f''(2.0)$ with Newton's forward difference cubic polynomial.

Example 2:

i) Complete the divided difference's table below:

i	x_i	$f(x_i)$	$f(x_i)^{[1]}$	$f(x_i)^{[2]}$	$f(x_i)^{[3]}$
0	2	14	22	(c)?	1
1	3	(a)?	(b)?	17	
2	6	234	151		
3	8	536			

ii) From (i), determine the Newton divided difference's polynomial.

iii) From (ii), estimate value of f when x = 2.5 and 6.4.

iv) Hence, estimate $f'(2)$, $f''(3)$ and $f'''(4)$.

Give your answer in 3 decimal places.