SSCE 2393 NUMERICAL METHODS

CHAPTER 6

NUMERICAL INTEGRATION

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Overview of Chapter 6

6.1 Introduction

6.2 Trapezoidal Rule

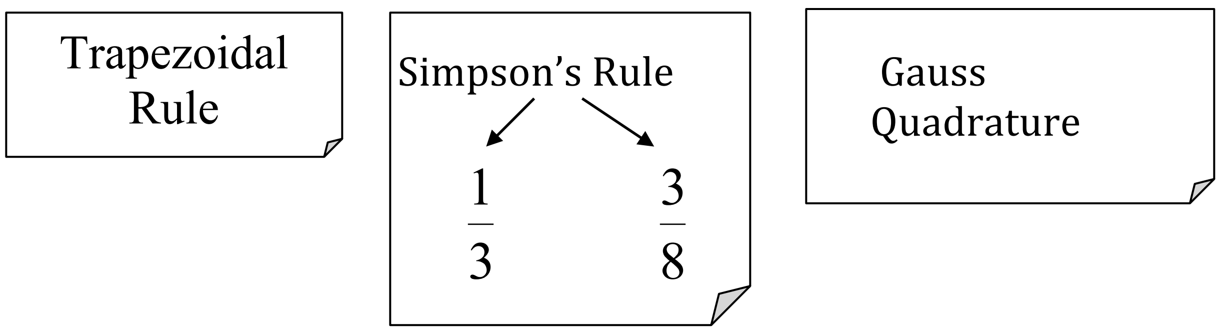
6.3 Simpson's Rule
a) 1/3 Simpson's rule
b) 3/8 Simpson's rule

6.4 Gauss Quadrature

a) 2 points
b) 3 points

6.1 Introduction

Aim: To solve $\int f(x)$ using numerical approach.
(useful for complicated function).



6.2 Trapezoidal Rule

- This method use linear polynomial interpolation, $P_1(x)$ to integrate a function $f(x)$.

- By Newton's forward-difference;

$$P_1(x) = f_0 + r\Delta f_0 \text{ where } r = \frac{x - x_0}{h}$$

Therefore:

$$\int_{a=x_0}^{b=x_1} f(x)\,dx \approx \int_{x_0}^{x_1} p_1(x)\,dx = \int_0^1 (f_0 + r\Delta f_0)(h\,dr)$$

$$= h\left[rf_0 + \frac{r^2}{2}\Delta f_0\right]_0^1 = h\left(f_0 + \frac{1}{2}\Delta f_0\right)$$

$$= h\left[f_0 + \frac{1}{2}(f_1 - f_0)\right] = h\left(\frac{1}{2}f_0 + \frac{1}{2}f_1\right)$$

$$= \frac{h}{2}(f_0 + f_1)$$

$$= \frac{h}{2}[f(a) + f(b)] \quad (\text{Area of Trapezium ABCD})$$

Hence:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a) + f(b)], \qquad h = b - a$$

If [a,b] is wide, then divide into *n* uniform sub-intervals.

And we have the general form of **composite Trapezoidal Rule:**

$$\int_a^b f(x)dx \approx \frac{h}{2}[(f_0 + f_n) + 2(f_1 + f_2 + \ldots + f_{n-1})], \qquad h = \frac{b-a}{n}$$

Example:

Use the composite Trapezoidal rule with the indicated values of *n* to approximate the following integrals. Do the calculation in 4 decimal places.

a) $\int_1^2 x \ln x \, dx \quad , \quad n = 4$

b) $\int_0^2 \frac{2}{x^2 + 4} \, dx \quad , \quad n = 6$

6.3 a) $\frac{1}{3}$ Simpson's Rule

This method use quadratic interpolation $P_2(x)$ (parabolic) to integrate $f(x)$.

By Newton's fwd difference:

$$p_2(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 \qquad , \; r = \frac{x - x_0}{h}$$

$$\int_{a=x_0}^{b=x_2} f(x)dx \approx \int_{x_0}^{x_2} p_2(x)dx = \int_0^2 \left(f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 \right)(h\,dr)$$

$$= h\left[rf_0 + \frac{r^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 f_0 \right]_0^2$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{2}\left(\frac{2}{3}\right)\Delta^2 f_0 \right]$$

$$= h\left(2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0) \right)$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2 \right]$$

Therefore, 1/3 Simpson's Rule is:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f(a+h) + f(b) \right], \qquad h = \frac{b-a}{2}$$

If [a,b] is wide, then divide the interval into n sub-intervals. ***n* must be even (i.e *n* = 2*m*).**

And we have the general form of **1/3 composite Simpson's rule:**

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[(f_0 + f_n) + 4(f_1 + f_3 + \ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots + f_{n-2})\right]$$

where $h = \dfrac{b-a}{n}$

Table:

i	x_i	$f(x_i)$		
		$f(a)$ and $f(b)$	$f(i = odd)$	$f(i = even)$
0 1 ⋮ N				

$$h = x_{i+1} - x_i \qquad n = \frac{b-a}{h}$$

Formula:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + f_n + 4\sum_{\substack{i=1\\ i\,\text{odd}}}^{n-1} f_i + 2\sum_{\substack{i=2\\ i\,\text{even}}}^{n-2} f_i\right]$$

Example 1:

Given:

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.1201	4.4257	6.0424	8.0301	10.4668

Approximate $\int_{1.8}^{2.6} f(x)\,dx$ using Simpson's Rule.

Example 2:

Use the Simpson's Rule with h = 0.25 to approximate $\int_{0}^{2} e^{x}\,dx$.

6.3 b) $\frac{3}{8}$ **Simpson's Rule**

This method use cubic interpolation $P_3(x)$ to integrate $f(x)$.

Solve the case when n = no. in factor 3 (i.e 3,6,9,12,18,..)

Table:

i	x_i	$f(x_i)$		
		$f(a)$ and $f(b)$	$f(i \neq triple)$	$f(i = triple)$
0				
1				
⋮				
N				

$h = x_{i+1} - x_i$ $\qquad\qquad n = \frac{b-a}{h}$

General formula for $\frac{3}{8}$ Simpson's Rule is:

$$\int_a^b f(x)\,dx \approx \frac{3}{8}h\left[\begin{array}{c}(f_0 + f_n) + 3(f_1 + f_2 + f_4 + f_5 + \cdots + f_{n-2} + f_{n-1}) \\ + 2(f_3 + f_6 + \cdots + f_{n-3})\end{array}\right]$$

Example 1:

Compute the area of cosine curve from 0 to $\frac{\pi}{2}$ using Simpson's Rule. Take $h = \frac{\pi}{18}$. Then, find the absolute error.

6.4 Gauss Quadrature

a) 2 points Gauss quadrature

Formula:

$$\int_a^b f(x)\ dx = \int_{-1}^{1} g(t)\ dt \approx g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

b) 3 points Gauss quadrature

Formula:

$$\int_a^b f(x)\ dx = \int_{-1}^{1} g(t)\ dt \approx \frac{5}{9} g\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} g(0) + \frac{5}{9} g\left(\sqrt{\frac{3}{5}}\right)$$

where

$$x = \frac{(b-a)t + (b+a)}{2}$$

Example:

1. Approximate $\int_1^3 \frac{x^3}{1+x^4}\,dx$ using

    a) 2-points Gauss Quadrature
    b) 3- points Gauss Quadrature

2. Evaluate $\int_1^2 \frac{e^x \sin x}{1+x^2}\,dx$ using

    a) 2-points Gauss Quadrature
    b) 3- points Gauss Quadrature