SSCE 2393 NUMERICAL METHODS

CHAPTER 7

EIGENVALUE PROBLEM

Farhana Johar, Department of Mathematical Sciences, Faculty of Science, UTM.
farhanajohar@utm.my

Overview of Chapter 7

7.1 Introduction:

- In Linear System, there are 2 methods of iteration.
- Iteration will converge when the matrix A of the system is strictly diagonally dominant.
- The characteristic of strictly diagonally dominant is directly related to the hidden value for the matrix which is known as eigenvalue.
- Importance of eigenvalue: to determine convergence of iteration. e.g. If magnitude < 1 for all eigenvalues, the iteration will converge and vice versa.
- Consider the following linear system $Av = \lambda v$, where λ is constant.
  - If value of $\lambda \neq 0$ and v is nonzero vector which satisfies the equation above, then:
    - λ is eigenvalue of A
    - v is the corresponding eigenvector
- $Av = \lambda v$ can be written as:

$$Av = \lambda Iv$$

$$(A - \lambda I)v = 0$$

where I is identity matrix.

Example:

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & -2 & -4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ -1 & 5-\lambda \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} 3 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & -2 & -4 \end{bmatrix} \lambda - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 & 1 \\ -1 & 3-\lambda & 1 \\ 1 & -2 & -4-\lambda \end{bmatrix}$$

Solve for $|A - \lambda I| = 0$ and $|B - \lambda I| = 0$

$$\det(A) = \lambda^2 - 8\lambda + 13$$

$$\det(B) = \lambda^3 + 2\lambda^2 + 16\lambda - 25$$

- Number of eigenvalues is depends on the size of the matrices:
  - matrix 2x2 => has 2 eigenvalues
  - matrix 3x3 => has 3 eigenvalues
  - matrix nxn => has n eigenvalues

- There are a few methods solving for λ
- The following theorem could assist the calculation of λ for matrix A with nxn size with eigenvalues, λ_i where $i = 1, 2, 3, .., n$

Theorem 1:

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

Theorem 2:

$$\prod_{i=1}^{n} \lambda_i = |A|$$

7.2 Gerschgorin Theorem

- Use to estimate λ value through determination of center and radius of circle.

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$ (Radius of a circle)

$$B_i = \{z \in C : |z - a_{ii}| \leq r_i\}$$ (Circle with radius r_i and center a_{ii})

Example:

Use Gerschgorin theorem to identify the position of all eigenvalue of matrix A:

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

From Gerschgorin theorem:

$$r_1 = \sum_{\substack{j=1 \\ j\neq 1}}^{3} |a_{1j}| = |a_{12}| + |a_{13}| = |-1| + |4| = 1 + 4 = 5$$

$$B_1 = \{z \in C : |z - a_{11}| \leq r_1\} = \{z \in C : |z - 1| \leq 5\}$$

$$r_2 = \sum_{\substack{j=1 \\ j\neq 2}}^{3} |a_{2j}| = |a_{21}| + |a_{23}| = |3| + |-1| = 3 + 1 = 4$$

$$B_2 = \{z \in C : |z - a_{22}| \leq r_2\} = \{z \in C : |z - 2| \leq 4\}$$

$$r_3 = \sum_{\substack{j=1 \\ j\neq 3}}^{3} |a_{3j}| = |a_{31}| + |a_{32}| = |2| + |1| = 2 + 1 = 3$$

$$B_3 = \{z \in C : |z - a_{33}| \leq r_3\} = \{z \in C : |z + 1| \leq 3\}$$

Therefore, all eigenvalue of A are $B_1 \cup B_2 \cup B_3$; lie in [-4,6].

7.3 Power Method

- Is an iterative method used to determine the ***dominant*** eigenvalue of a matrix – that is, the eigenvalue with the ***largest magnitude***.

Definition:

If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of matrix A and that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|,$$

Hence, λ_1 is the dominant eigenvalue.

Power Method Formula:

$$v^{(k+1)} = \frac{1}{m_{k+1}} A v^{(k)}, \qquad k = 0, 1, 2, ...$$

k	$\left[v^{(k)}\right]^T$	$\left[Av^{(k)}\right]^T$	m_{k+1}
0	$v^{(0)}$ given		
1			
⋮			
n			

use formula

m_{k+1} = largest eigenvalue

(in absolute value)

Stopping criterion: $\left|m_{k+1} - m_k\right| < \varepsilon$

Example:

Use Power Method to determine the dominant eigenvalue and eigenvector of the following matrices:

a)

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Let $v^{(0)} = (0,1,0)$.

b)

$$A = \begin{bmatrix} 10 & 4 & 1 \\ 1 & 10 & 0.5 \\ 1.5 & -3 & 20 \end{bmatrix}.$$

Let $v^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

Do the calculation in 4 decimal places.

7.4 Shifted Power Method

- Use to find smallest eigenvalue, λ_n.

1. We need a shifted factor, *p*

$$p = \lambda_1 \qquad \lambda_1 \text{ - dominant eigenvalue}$$

2. Find matrix B:

$$B = A - pI \qquad I = \text{Identity matrix}$$

3. Use **power method** to get λ_B (the dominant eigenvalue for matrix B)

4. Find λ_n :

$$\lambda_n = \lambda_B + p$$

5. The associated eigenvector to λ_n is given by V_n.

$$V_n = V_B$$

Important properties:

1. $\lambda_1 + \lambda_2 + .. + \lambda_n = a_{11} + a_{22} + .. + a_{nn}$
2. $\lambda_1 \cdot \lambda_2 \cdot\cdot \lambda_n = |A|$

Example:

1. Given: $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$

a) Find the dominant eigenvalue and associated eigenvector of matrix A using power method. ($v^{(0)} = (1,0,0)$)

b) Determine the absolutely smallest eigenvalue of A and associated eigenvector using shifted power method. ($v^{(0)} = (1,0,0)$)

Do the calculation in 2 decimal places and $\varepsilon = 0.01$.

2. Find the absolutely smallest eigenvalue and associated eigenvector of the following matrices using shifted power method.

a)

$$A = \begin{bmatrix} 10 & 4 & 1 \\ 1 & 10 & 0.5 \\ 1.5 & -3 & 20 \end{bmatrix}.$$

Let $v^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\lambda_{\text{dominant}} = 20$

b)

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Let $v^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\lambda_{\text{dominant}} = 5.236$

Example from Final exam: (Gersgorin, Power method, shifted power method.)

Given $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

a) By using the Gerschorin's theorem, show that all the eigen values are either positive or negative sign.

b) Compute the dominant eigen value λ_1 and the corresponding eigen vector v for matrix A by using the power method with $v^{[0]} = (1,0,0)^T$ and stop the calculation when $|m_{k+1} - m_k| < 0.05$.

c) Determine the smallest in magnitude the eigenvalue λ_3 of **A** and its associated eigenvector $\mathbf{v}_3$ by the shifted power method with shifting factor $p = \lambda_1$. Start your iteration with $\mathbf{v}^{(0)} = (1,1,1)^T$ and stop the iteration when $|m_k - m_{k-1}| < 0.05$.

d) Calculate the eigen value λ_2 for matrix *A* base on the results in (b) and (c).