SSCE 2393 NUMERICAL METHODS

CHAPTER 8

ORDINARY DIFFERENTIAL EQUATION (ODE)

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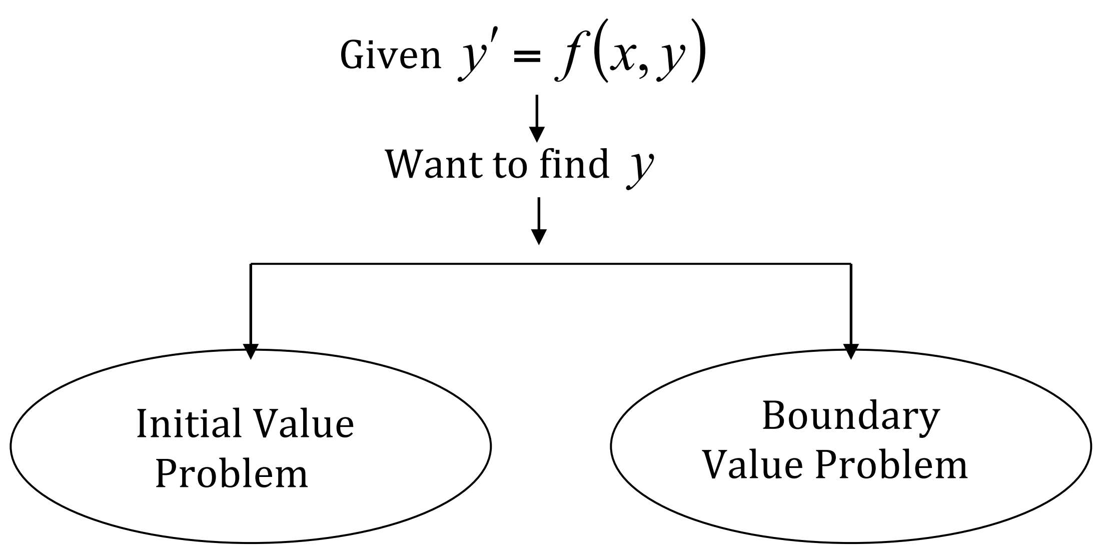
Overview of Chapter 8

8.1 Introduction to the chapter

8.2 Initial Value Problem (IVP)

8.3 Boundary Value Problem (BVP)

8.1 Introduction to ODE



8.1 Initial Value Problem (IVP)

Taylor's Series Method

1. Euler's Method
2. Taylor's Mtd order 2

Runge-Kutta Method

1. R-K order 2
2. R-K order 4

8.1.1 Taylor's Method

a) <u>Euler's Method (Taylor's Method order 1)</u>

$$\boxed{y(x_{i+1}) = y(x_i) + hy'(x_i)} + \frac{h^2}{2!}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i)$$

$$y_{i+1} = y_i + hy_i'$$

i	x_i	y_i	$y(x_i)$ (exact)	$\lvert$error$\rvert$
0		Given y_0		
1				
⋮				
N				

Exact sol. for $y(x)$

Absolute error (if needed)

Example: (Do all calculation in 4 dcp)

1. Use Euler's Method to approximate the solutions for the following IVPs:

a) $y' = 1 + \frac{y}{x}$, $y(1) = 2$ for $1 \le x \le 2$ with $h = 0.25$

Given the actual solution is $y = x\ln x + 2x$. Compute the absolute error.

b) $\frac{dy}{dt} = \cos 2t + \sin 3t,\ y(0) = 1$ for $0 \le t \le 1$ with $h = 0.25$

Given the actual solution is $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

Compute the absolute error.

2. The equation for the upward velocity *v* for a rocket is given by:

$$\frac{dv}{dt} = \frac{5000 - 0.1v^2}{300 - 10t} - g \ ; \quad v = 0 \text{ at } t = 0 \text{ where}$$

v in m/s, *t* is the time in seconds, and *g* is 9.81m/s^2.

Use time steps of 2 s to solve for *v* by the Euler method, and produce a table of (*t*,*v*) values in time increments of 2 s up to 10 s.

3. If water is drained from a vertical cylindrical tank by opening a valve at the base, the water will flow fast when the tank is full and slow down as it continous to drain. The rate at which the water level drops is:

$$\frac{dy}{dt} = -k\sqrt{y}$$

where k is a constant. The depth of the water, y is measured in feet and time, t in minutes.

If $k = 0.1$, determine how long it takes the tank to drain if the water level is initially 9ft.

Solve by applying Euler's method and use a step of 4 minutes.

b) Taylor's Method order 2

$$\boxed{y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i)} + \cdots + \frac{h^n}{n!}y^{(n)}(x_i)$$

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i''$$

i	x_i	y_i	$y(x_i)$ (exact)	|error|
0		Given y_0		
1				
⋮				
n				

Example:

1. Use Taylor's Method of order 2 to approximate the solutions for the following IVPs:

a) $y' = 1 + (x - y)$, $y(2) = 1$ for $2 \le x \le 3$ with $h = 0.5$

Given the actual solution is $y = x - e^{(2-x)}$. Compute the absolute error.

b) $y' = te^{3t} - 2y$, $y(0) = 0$ for $0 \le t \le 3$ with $h = 0.5$

8.1.2 Runge-Kutta Method

a) R-K order 2

i) Improved Euler's Method

$$y_{i+1} = y_i + \frac{1}{2}k_1 + \frac{1}{2}k_2$$

where

$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf(x_i + h, y_i + k_1)$$

i	x_i	y_i	k_1	k_2	Exact	$\lvert error \rvert$
0		Given y_0				
1						
⋮						
N						

Example: (Calculation in 4 dcp)

1. Employ Improved Euler's method to solve the following problem. Compute the absolute error.

a) $\dfrac{dy}{dt} + 4yt = 6t, \quad y(0) = 1, \; 0 \le t \le 1, \; y(t) = -\dfrac{1}{2}e^{-2t^2} + \dfrac{3}{2}, \; h = 0.2$

b) $y' - y = 2e^{4x}, \quad y(0) = -3, \; 0 \le x \le 1, \; y(x) = \dfrac{2}{3}e^{4x} + \dfrac{11}{3}e^{x} \;, h = 0.2$

ii) Midpoint Method

$$y_{i+1} = y_i + k_2$$

where

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

iii) Heun's Method

$$y_{i+1} = y_i + \frac{1}{4}\left(k_1 + 3k_2\right)$$

where

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{2}{3}h, y_i + \frac{2}{3}k_1)$$

Example:

Solve the following first order initial value problem at $x = 0(0.2)0.6$ by mid-point method;

$$2y'+3y = e^{2x}, \qquad y(0) = 1$$

Given the exact solution is $y = \frac{1}{7}e^{2x} + \frac{6}{7}e^{-\frac{3x}{2}}$. Determine the absolute error and do the calculation in 4 decimal places.

b) R-K order 4

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$
$$k_3 = hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$
$$k_4 = hf(x_i + h, y_i + k_3)$$

i	x_i	y_i	k_1	k_2	k_3	k_4	Exact	$\lvert error \rvert$
0		**Given**						
1								
⋮								
n								

Example:

Solve the following first order initial value problem at $x = 0(0.2)0.4$ by using fourth-order Runge-Kutta method;

$$2y' + 3y = e^{2x}, \qquad y(0) = 1$$

Given the exact solution is $y = \frac{1}{7}e^{2x} + \frac{6}{7}e^{-\frac{3x}{2}}$. Determine the absolute error and do the calculation in 4 decimal places.

8.2 Boundary Value Problem

Consider 2 types of boundary condition;

Without Differentiation	With Differentiation

Solve these 2 types of problem by using finite difference method.

$$y'_i \approx \frac{y_{i+1} - y_{i-1}}{2h} \qquad y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

a) BVP Without Differentiation

$$y'' + p(x)y' + q(x)y = r(x), \qquad y(a) = \alpha, \quad y(b) = \beta$$

Example:

1. Solve the following boundary value problem at $x = 1(0.2)2$ by using finite difference method;

$$y'' + \left(\frac{1}{x}\right)y' - \left(\frac{1}{x^2}\right)y = 3 \qquad y(1) = 2, \quad y(2) = 3$$

2. Solve the following boundary value problem;

$$y'' - \left(1 - \frac{x}{5}\right)y = x$$
$$y(1) = 2, y(3) = -1, h = 0.5$$

b) BVP With Differentiation

$$y'' + p(x)y' + q(x)y = r(x),$$

$$ky(a) + ly' = \alpha, \quad l \neq 0$$
$$my(b) + ny'(b) = \beta, \qquad n \neq 0$$

Example:

1. Solve the boundary value problem at $x = 0(0.2)1$ by using finite difference method.

$$y'' + 2xy' - 3y = -6e^{-x}\left(1 + x^2\right), \quad y'(0) + 0y(0) = 3, \; y'(1) = 0$$

(Exact solution: $y(x) = 3xe^{-x}$)

2. Solve the BVP;

$$y'' = y$$
$$y'(1) = 1.175, y'(3) = 10.018, h = 0.5.$$