SSCE 2393 NUMERICAL METHODS

CHAPTER 9

PARTIAL DIFFERENTIAL EQUATIONS

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PARTIAL DIFFERENTIAL EQUATION

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

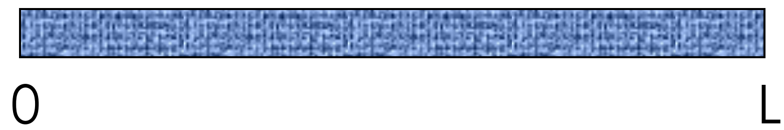
$B^2-4AC=0$	$B^2-4AC>0$	$B^2-4AC<0$
↓	↓	↓
Parabolic Equation	Hyperbolic Equation	Elliptic Equation

Solution Steps:

1. Draw the grid solutions and consider the value of boundary and initial conditions.
2. Write PDE in the form of finite-difference .
3. Calculate $u_{i,j}$ level by level (molecule graph).
4. Find the answer through the solution of linear equation system.

Parabolic Equation

Focus: Heat Equation



Temperature distribution: $u(x,t)$

Heat Equation: $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \quad 0 < x < L, t > 0$

Boundary conditions : $\begin{aligned} u(0,t) &= c_1 \\ u(L,t) &= c_2 \end{aligned}$

Initial condition : $u(x,0) = f(x)$

Method

→ Explicit Finite-Difference method

<u>Explicit Finite-Difference Method</u>

$$u(x,t) = u(x_i, t_j)$$
$$= u_{i,j}$$

Finite difference formula for this method:

$$\frac{\partial u_{i,j}}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$ (forward difference)

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$ (central difference)

$k = t_{i+1} - t_i = \Delta t$ and $h = x_{i+1} - x_i = \Delta x$

Hence,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

e.g.

Given the heat equation:

$$\frac{\partial u}{\partial t} - \frac{4}{\pi^2}\frac{\partial^2 u}{\partial x^2} = 0$$

with

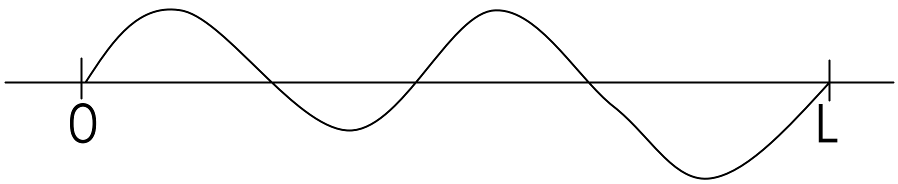
$$u(0,t) = u(4,t) = 0 \qquad t > 0$$

$$u(x,0) = \left(\sin\frac{\pi x}{4}\right)\left(1 + \cos\frac{\pi x}{4}\right) \qquad 0 \le x \le 4$$

take $h = 1.0$ and $k = 0.04$, solve the heat equation up to second level.

Hyperbolic Equations

Focus: Wave Equations



Displacement distribution : $u(x,t)$

Wave equation : $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$$0 < x < L, t > 0$$

Boundary conditions : $u(0,t) = 0$, $u(L,t) = 0$

Initial conditions : $u(x,0) = f(x)$ → initial displacement

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$ → initial velocity

method → finite difference

Finite Difference Method

$$u(x,t) = u(x_i, t_j)$$
$$= u_{i,j}$$

Finite difference formula:

$$\frac{\partial^2 u_{i,j}}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Initial conditions (velocity)

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

} Consider this condition at 1st level

$k = t_{i+1} - t_i = \Delta t$ and $h = x_{i+1} - x_i = \Delta x$

Thus,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\Downarrow$$

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

e.g.

Given the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{\pi}\frac{\partial^2 u}{\partial x^2} = 0 \qquad\qquad 0 < x < 2$$

where

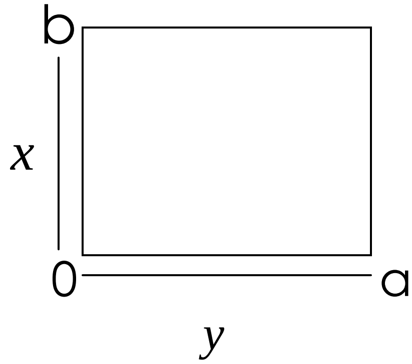
$$u(0,t) = u(2,t) = 0 \qquad\qquad t > 0$$

$$\left.\begin{array}{l} u(x,0) = \sin(\pi x) \\ u_t(x,0) = \cos(2\pi x) \end{array}\right\} \qquad\qquad 0 \le x \le 2$$

take $\Delta x = k = 0.1$, solve this equation up to the second level.

Elliptic Equations

Focus: Poisson Equation and Laplace's Equation



Steady-state temperature distribution : $u(x, y)$

Poisson Equation : $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f(x, y)$$

$$0 < x < a, 0 < y < b$$

If; $f(x, y) = 0$ then,

Laplace's equation : $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

$$0 < x < a,\ 0 < y < b$$

Boundary conditions :

$$\left.\begin{aligned} u(0, y) &= f_1(y) \\ u(a, y) &= f_2(y) \end{aligned}\right\} 0 \le y \le b$$

$$\left.\begin{aligned} u(x, 0) &= g_1(x) \\ u(x, b) &= g_2(x) \end{aligned}\right\} 0 \le x \le a$$

Method → finite difference

<u>Finite Difference Method</u>

$$u(x,t) = u(x_i, t_j)$$
$$= u_{i,j}$$

Finite Difference Formula:

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$\frac{\partial^2 u_{i,j}}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Thus,

Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

$$\Downarrow$$

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = f(x_i, y_j)$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\Downarrow$$

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}=0$$

e.g. (Poisson's Equation)

Use finite difference method to find the approximate solutions for this equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(x^2+y^2\right)e^{xy} \quad 0<x<2, 0<y<1$$

with $(0,y)=u(x,0)=1$, $u(2,y)=e^{2y}$ and $u(x,1)=e^{x}$.
Take $\Delta x=h=1.0$ and $\Delta y=k=0.25$.

e.g. (Laplace's Equation)

Use finite difference to find the linear equation system for

$$u_{xx} + u_{yy} = 0 \qquad\qquad 0 < x < 1.5, 0 < y < 1.5$$

with $u(0, y) = u(x,0) = 0$, $u(1.5, y) = u(x,1.5) = 2$. Take $h = k = 0.5$.