Chapter 1: Multivariable Functions

1.1 Functions of two variables

Definition:

A function f of two real variables, *x*, and *y*, is a rule that assigns a unique real number $f(x,y)$ to each point (x,y) in some set *D* of the *xy* –plane.

The **domain** of a function is a set of points at which the function is defined.

Example 1.1:

Let $f(x,y)=3x^2\sqrt{y}-1$. Find $f(1,4)$, $f(0,9)$, and the natural domain of f.

Example 1.2:

Sketch the natural domain of the function

$f(x,y)=\ln(x^2-y)$

Example 1.3

Find the domain and the range of the function $f(x,y)=\dfrac{1}{xy}$

Definition: *A level curve* of a function $f(x, y)$ is the curve $f(x, y) = c$ in the *xy*-plane. It consists of the points (x, y) where the function has the value *c*.

Example:

Sketch the level curves of $z = f(x, y) = x^2 + y^2 - 9$ for $z = -9, -5, 0, 7$, then sketch the graph of the function.

Common surfaces

1. Paraboloid : $cz = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

2. Cone : $cz = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$

3. Double cones : $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

4. Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

5. Hyperboloid of one sheet : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

6. Hyperboloid of two sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

7. Hyperbolic Paraboloid : $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz, \quad c > 0$

1.2 Functions with three variables

Definition:

A function f of three real variables, *x*, *y*, and *z*, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set *D* of three-dimensional space.

Example:

Find the domain of the following functions:

(i) $f(x, y, z) = xy \ln z$

(ii) $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$

Level Surfaces

Definition: A *level surface* of $f(x,y,z)$ is the surface $f(x,y,z)=c$ in *xyz*-space where the function has a constant value.

Example:

If $w=f(x,y,z)=z-\sqrt{x^2+y^2}=c$, sketch the level surfaces of f, for $c=-1,0,1,2$.