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SSH 1033 MATHEMATICAL METHODS II **TUTORIAL 4**

1. State where the following power series centered.

(a) $\sum_{n=0}^{\infty} nx^n$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \, 1 \cdot 3 \cdots (2n-1)}{2^n \, n!} x^n$.

(c) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n}}{(2n)!}$.

2. Write an equivalent series with the index of summation beginning at $n = 1$.

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(b) $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$.

(c) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

3. Find the radius of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$.

(b) $\sum_{n=0}^{\infty} (2x)^n$.

(c) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$.

(e) $\sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$.

(f) $\sum_{n=0}^{\infty} \frac{(2n)! \, x^{2n}}{n!}$.

4. Find the interval of convergence of the following power series. Be sure to include a check for convergence at the endpoints of the interval.

(a) $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$.

(b) $\sum_{n=0}^{\infty} 3^n x^n$.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.

(e) $\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$.

(f) $\sum_{n=1}^{\infty} \frac{5^n}{n^2} x^n$.

(g) $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$.

(h) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

(i) $\sum_{n=0}^{\infty} \frac{(-2)^n x^{n+1}}{n+1}$.

(j) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{n^3} (x-2)^n$.

(k) $\sum_{n=1}^{\infty}(-1)^n \frac{(x+1)^{2n+1}}{n^2+4}$.

(l) $\sum_{n=1}^{\infty} \frac{(\ln n)(x-3)^n}{n}$.

(m) $\sum_{n=0}^{\infty} \frac{\pi^n (x-1)^{2n}}{(2n+1)!}$.

(n) $\sum_{n=0}^{\infty} \frac{(2x-3)^n}{4^{2n}}$.

5. Write the first four terms of the series and find the radius of convergence.

(a) $\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 4\cdot 7\cdot (3n-2)} x^n$.

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 3\cdot 5\cdot (2n-1)} x^{2n+1}$.

(c) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots (2n-1)}{(2n-2)!} x^n$.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3\cdot 7\cdot 11\cdots (4n-1)(x-3)^n}{4^n}$.

(e) $\sum_{n=1}^{\infty} \frac{n!\,(x+1)^n}{1\cdot 3\cdot 5\cdots (2n-1)}$.

6. Use the definition to find the Taylor series (centered at c) for the function.

(a) $f(x)=e^{2x}$; $c=3$.

(b) $f(x)=e^{3x}$; $c=0$.

(c) $f(x)=\frac{1}{x}$; $c=-1$.

(d) $f(x)=\ln x$; $c=1$.

(e) $f(x)=\cos x$; $c=\frac{\pi}{2}$.

(f) $f(x)=\sin \pi x$; $c=\frac{1}{2}$.

7. Find the Maclaurin series for the given function. Express your answer in sigma notation.

(a) $f(x)=e^{-x}$.

(b) $f(x)=e^{ax}$.

(c) $f(x)=\frac{1}{1+x}$.

(d) $f(x)=xe^{x}$.

(e) $f(x)=\ln(1+x)$.

(f) $f(x)=\sin \pi x$.

(g) $f(x)=\cos\left(\frac{x}{2}\right)$.

(h) $f(x)=\sinh x$.

8. Find the Maclaurin series for the function. Use the table of power series for elementary functions.

(a) $f(x)=e^{-3x}$.

(b) $f(x)=e^{x^2}$.

(c) $f(x)=\sin 3x$.

(d) $f(x)=\cos 4x$.

(e) $f(x)=2\sin x^3$.

(f) $f(x)=\cos^2 x$.

9. Find the intervals of convergence of (i) $f(x)$, (ii) $f'(x)$, and (iii) $\int f(x)dx$. Include a check for convergence at the endpoints of the interval.

(a) $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$.

(b) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n\,5^n}$.

(c) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$.

(d) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$.

10. Show that the Maclaurin series for $f(x) = \cos x$ is given by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

and determine its radius of convergence.
Hence, use this result to find the power series and interval of convergence for the following functions.

(a) $g(x) = \cos\sqrt{x}$.

(b) $h(x) = \sin^2 x$.

11. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the interval of convergence of f.
(b) Show that $f'(x) = f(x)$.
(c) Show that $f(0) = 1$.
(d) Identify the functions f.

12. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(a) Find the intervals of convergence of f and g.
(b) Show that $f'(x) = g(x)$.
(c) Show that $g'(x) = -f(x)$.
(d) Identify the functions f and g.

13. Expand the integrand the the first four nonzero terms and use it to approximate the value of the integral.

(a) $\int_0^1 \sin x^2\, dx$.

(b) $\int_0^1 \cos\sqrt{x}\, dx$.

(c) $\int_0^{0.1} \frac{\sin x}{x}\, dx$.

(d) $\int_0^{1/2} \frac{dx}{1+x^4}$.

(e) $\int_0^{1/2} \tan^{-1} 2x^2\, dx$.

(f) $\int_0^{0.1} e^{-x^3}\, dx$.

14. Obtain the familiar result, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, by finding a power series for $\frac{\sin x}{x}$ and taking the limit term by term.

15. Use the method of Exercise 14 to find the limits.

(a) $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$. (b) $\lim_{x\to 0} \frac{\ln\sqrt{1+x} - \sin 2x}{x}$.

16. Find a power series for $\frac{1-\cos 3x}{x^2}$ and use it to evaluate $\lim_{x\to 0} \frac{1-\cos 3x}{x^2}$.

17. Find a power series for $\frac{\ln(1-2x)}{x}$ and use it to evaluate $\lim_{x\to 0} \frac{\ln(1-2x)}{x}$.

18. Show that the function represented by the power series is a solution of the differential equation.

(a) $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $\frac{d^2y}{dx^2} - y = 0.$

(b) $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\frac{d^2y}{dx^2} + y = 0.$

(c) $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!}$, $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0.$

19. The Bessel function of order 0 is defined by the power series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\,(n!)^2}.$$

(a) Show that the series converges for all x.

(b) Show that $J_0(x)$ is a solution of the differential equation

$$x\frac{d^2J_0}{dx^2} + \frac{dJ_0}{dx} + xJ_0 = 0.$$