CHAPTER 3 : SERIES

3.1 Power Series

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \ldots + a_n (x-a)^n + \ldots$$

in which the center a and the coefficients $a_0, a_1, a_{2,} \ldots, a_n, \ldots$ are constants.

Expansion of Exponent Function

The power series of the exponent function can be written as

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

The expansion is true for all values of x. In general,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

Example (1):

Given

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots + \frac{1}{n!}x^n + \dots$$

Write down the first five terms of the expansion of the following functions

(a) e^{2x}

(b) e^{x-1}

Example (2):

Write down the first five terms on the expansion of the function, $(1+x)^2 e^{-x}$ in the form of power series.

Expansion of Logarithmic Function

The expansion of logarithmic function can be written as

$$\ln(1+\mathrm{x}) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \dots$$

The series converges for $-1 < x \le 1$. Thus the series $\ln(1+x)$ is valid for $-1 < x \le 1$.

By assuming x with $-x$, we obtain

$$\ln(1-\mathrm{x}) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6 - \frac{1}{7}x^7 - \dots$$

Thus, this result is true for $-1 < -x \leq 1$ or $-1 \leq x < 1$.

Example (3):

Write down the first five terms of the expansion of the following functions

(a) $\ln(1+3x)$

(b) $3\ln(1-2x^2)(1+3x)$

Example (4):

Find the first four terms of the expansion of the function, $(1+x)^2 \ln(1+2x)^3$.

Expansion of Trigonometric Function

The power series for trigonometric functions can be written as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Both series are valid for all values of x.

Example (5):

Given

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Find the expansion of $\cos(2x)$ and $\cos(3x)$. Hence, by using an appropriate trigonometric identity find the first four terms of the expansion of the following functions:

(a) $\sin^2(x)$ (b) $\cos^3(x)$

3.2 The Taylor and the Maclaurin Series

Definition 5.9 (TAYLOR AND MACLAURIN SERIES)

If $f(x)$ has a derivatives of all orders at $x = a$, then we call the series as **Taylor's Series** for $f(x)$ about $x = a$ and is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^r}{r!}f^r(a) + \cdots.$$

or

$$f(x+a) = f(a) + x\,f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^r}{r!}f^r(a) + \cdots.$$

In the special case where $a = 0$, this series becomes the **Maclaurin Series** for $f(x)$ and is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x}{r!}f^r(0) + \cdots \quad \diamond$$

Example (1):

Obtain the Taylor series for $f(x) = 3x^2 - 6x + 5$ around the point $x = 1$.

Example (2):

Obtain Maclaurin series expansion for the first four terms of e^x and five terms of $\sin x$. Hence, deduct that Maclaurin series for $e^x \sin x$ is given by

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

Example (3):

Use Taylor's theorem to obtain a series expansion of first five terms for $\cos\left(x + \frac{\pi}{3}\right)$.

Hence find $\cos 62^\circ$ correct to 4 dcp.

Example (4):

If $y = \ln\cos x$, show that

$$\frac{d^2 y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$$

Hence, by differentiating the above expression several times, obtain the Maclaurin's series of $y = \ln \cos x$ in the ascending power of x up to the term containing x^4.

Finding Limits with Taylor Series and Maclaurin Series.

Example (5):

Find $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

Example (6):

Evaluate $\lim_{x\to 0} \dfrac{x^2 + 2\cos x - 2}{3x^4}$.

Evaluating Definite Integrals with Taylor Series and Maclaurin Series.

Example (7):

Use Maclaurin series to approximate the following definite integral.

(a) $\int_0^1 e^{-x^2}\,dx$

(b) $\int_0^1 x\cos(x^3)\,dx$