Universiti Teknologi Malaysia
Department of Mathematical Sciences
Semester 1, 2016/17
Date: 13 **October** 2016

SSCM1033 Mathematical Method II   Test 1(15%)   Time: $1\frac{1}{4}$hr

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ANSWER ALL QUESTIONS

1. Determine whether the sequence converges or diverges. If it is converges, find the limit.

   (a) $\{\frac{n+1}{2n^2}\}$

   (b) $\{\frac{e^{2n}}{n}\}$

   (c) $\{\sqrt[n]{2^{1+3n}}\}$

   [7 marks]

2. Express the sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$ in the notation of $\{a_n\}_{n=1}^{+\infty}$. Hence, show its limit converges.

   [2 marks]

3. Show that the following series is a telescoping series. Hence, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

   [6 marks]

4. Determine whether the series converges or diverges by using any appropriate test:

$$\sum_{n=1}^{\infty} \frac{n^n}{2^n}$$

   [5 marks]

5. Show that the given alternating series is converges:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{7}{n^3+1}$$

   [4 marks]

6. Find the first 3 terms of the Taylor series for the following function:

$$f(x) = \sin(\pi x)$$

   centred at $a = 0.5$.

   [6 marks]

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1. a) $\left\{ \frac{n+1}{2n^2} \right\} \rightarrow \lim_{n\to\infty} \frac{n+1}{2n^2} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{1}{4n} = 0$ converges $\boxed{2}$

b) $\left\{ \frac{e^{2n}}{n} \right\} \rightarrow \lim_{n\to\infty} \frac{e^{2n}}{n} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} 2e^{2n} = \infty$ diverges $\boxed{2}$

c) $\left\{ \sqrt[n]{2^{1+3n}} \right\} \rightarrow \lim_{n\to\infty} 2^{\frac{1+3n}{n}} = \lim_{n\to\infty} 2^3 \cdot 2^{1/n}$

$= \lim_{n\to\infty} 8 \cdot 2^{1/n} = 8$ $\boxed{3}$

(Converges)

$\boxed{7}$

2. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots = \left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty} = \{a_n\}$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$ (converges) $\boxed{2}$

3. $\sum \frac{1}{n^2+n}$

$a_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ ← partial fraction

$\Rightarrow 1 = A(n+1) + Bn$
$= An + A + Bn$ $\Big\}$ $A = 1$, $A + B = 0$
$A = -B$
$\Rightarrow B = -1$

$\therefore a_n = \frac{1}{n} - \frac{1}{n+1} = b_n - b_{n+1}$

$b_n = \frac{1}{n}$. $\therefore$ the series is telerscoping series

Thus, $\sum a_n = b_1 - \lim_{n\to\infty} b_n$

$= \frac{1}{1} - \lim_{n\to\infty} \frac{1}{n}$

$= 1$ ※

$\boxed{6}$

-t) $\sum_{n=1}^{\infty} \frac{n^n}{2^n}$, $a_n = \frac{n^n}{2^n}$

Using Ratio Test : $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{2^{(n+1)}} \cdot \frac{2^n}{n^n}$

$= \lim_{n\to\infty} \frac{1}{2} \frac{(n+1)^n(n+1)}{n^n}$

$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^n (n+1)$

$= \frac{1}{2} \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n}_{A} \cdot \lim_{n\to\infty} n+1$

Root Test :

$\lim_{n\to\infty} \left(\left(\frac{n}{2}\right)^n\right)^{\frac{1}{n}}$

$\lim_{n\to\infty} \frac{n}{2} = \infty$

(divergence)

$A = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$= e^1$ (using elimit)

$= \frac{1}{2} e^1 \cdot \infty = \infty$ (divergence) ※.

5

OR

$\lim_{n\to\infty} y = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$y = \left(1+\frac{1}{n}\right)^n$

$\ln y = n\left[\ln\left(1+\frac{1}{n}\right)\right]$

$= \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}$

$\therefore \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n} = \frac{0}{0}$

$= \lim_{n\to\infty} \frac{\frac{1}{1+1/n} \cdot \cancel{-\frac{1}{n^2}}}{\cancel{-\frac{1}{n^2}}} = 1$

$\therefore \lim_{n\to\infty} y = e^1$ ※.

5) $\sum (-1)^{n-1} \dfrac{7}{n^3+1}$

$$f(x) = \frac{7}{x^3+1} = 7(x^3+1)^{-1}$$

$$f'(x) = -7(x^3+1)^{-2} \cdot 3x^2$$

$$= -\frac{21x^2}{(x^3+1)^2} < 0 \text{ (decreasing)}$$

$$\lim_{n\to\infty} \frac{7}{n^3+1} = 0$$

$\therefore$ the series is converges ※.

$\cancel{8}$ $\boxed{4}$

6) $f(x) = \sin(\pi x)$, $a = 0.5$ $\Rightarrow f(1/2) = 1$

$f'(x) = \pi\cos(\pi x)$, $f'(1/2) = 0$

$f''(x) = -\pi^2\sin(\pi x)$, $f''(1/2) = -\pi^2$

$f'''(x) = -\pi^3\cos(\pi x)$, $f'''(1/2) = 0$

$f^{(4)}(x) = \pi^4\sin(\pi x)$, $f^{(4)}(1/2) = \pi^4$

Taylor series at $x = 1/2$:

$$f(x) = f(\tfrac{1}{2}) + (x-\tfrac{1}{2})f'(\tfrac{1}{2}) + \frac{(x-\frac{1}{2})^2 f''(\frac{1}{2})}{2!} + \frac{(x-1/2)^3 f'''(1/2)}{3!} + \frac{(x-1/2)^4 f^{(4)}(1/2)}{4!}$$

$$f(x) = 1 - \frac{\pi^2}{2!}(x-1/2)^2 + \frac{\pi^4(x-1/2)^4}{4!} + \dots \text{ ※}$$

$\boxed{\text{or}}$

$$f(x-1/2) = f(1/2) + xf'(1/2) + \frac{x^2 f''(1/2)}{2!} + \frac{x^3}{3!}f'''(1/2) + \frac{x^4}{4!}f^{(4)}(1/2) + \dots$$

$$= 1 + 0 - \frac{\pi^2 x^2}{2!} + 0 + \frac{\pi^4 x^4}{4!}$$

$$= 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{4!}x^4 + \dots \text{ ※}$$

$\boxed{6}$