Universiti Teknologi Malaysia
Faculty of Science
Department of Mathematical Sciences
Semester 2 Session 2015/2016

Test 1 (15%) **Time: 75 minutes**
SSCM 1033 Mathematical Methods II

Answer all questions.

1. Express the sequence

$$(\sqrt{2}-\sqrt{3}),(\sqrt{3}-\sqrt{4}),(\sqrt{4}-\sqrt{5}),\cdots$$

in the notation of $\{a_n\}_{n=1}^{\infty}$. Hence, show its limit converges to 0.

[3 marks]

2. Find the limit of $\left\{n\sin\frac{1}{n}\right\}_{n=1}^{\infty}$.

[4 marks]

3. By using ratio of successive terms method, prove that

$$\left\{\frac{2^n}{3^{2n}}\right\}_{n=1}^{\infty}$$

is a decreasing sequence.

[3 marks]

4. Determine whether the series

$$\sum_{n=0}^{\infty}\frac{2^{n+2}}{5^n}$$

converges or diverges. Find its sum if the series converges.

[4 marks]

5. Show that the series

$$\sum_{n=1}^{\infty}\frac{3}{2n(n+1)}$$

is a telescoping series. Hence, determine whether the series converges or diverges.

[6 marks]

6. Determine the convergence of the series

$$\frac{1}{3}+\frac{1}{6}+\frac{1}{11}+\cdots+\frac{1}{n^2+2}+\cdots$$

by using any appropriate test.

[4 marks]

7. Find the interval of convergence for the series

$$\sum_{n=1}^{\infty}\left(-\frac{x^n}{n}\right).$$

[6 marks]

1.

$$
\begin{aligned}
\{a_n\}_{n=1}^{\infty} &= \left\{\sqrt{n+1}-\sqrt{n+2}\right\}_{n=1}^{\infty} && [\mathbf{B1}]\\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n+2})\\
&= \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n+2}) \times \frac{(\sqrt{n+1}+\sqrt{n+2})}{(\sqrt{n+1}+\sqrt{n+2})} && [\mathbf{M1}]\\
&= \lim_{n\to\infty}\frac{(n+1)-(n+2)}{(\sqrt{n+1}+\sqrt{n+2})}\\
&= \lim_{n\to\infty}\frac{-1}{(\sqrt{n+1}+\sqrt{n+2})} && [\mathbf{A1}]\\
&= 0\\
&\text{(shown).}
\end{aligned}
$$

2.

$$
\begin{aligned}
\lim_{n\to\infty}\left(n\sin\frac{1}{n}\right) &= \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) && [\mathbf{M1}]\\
&= \lim_{n\to\infty}\left(\frac{\left(-\frac{1}{n^2}\right)\cos\frac{1}{n}}{\left(-\frac{1}{n^2}\right)}\right) && [\mathbf{M1}],[\mathbf{A1}]\\
&= \lim_{n\to\infty}\left(\cos\frac{1}{n}\right)\\
&= 1. && [\mathbf{A1}]
\end{aligned}
$$

3.

$$
\begin{aligned}
a_n = \frac{2^n}{3^{2n}}, \quad a_{n+1} &= \frac{2^{n+1}}{3^{2n+2}} && [\mathbf{B1}]\\
\frac{a_{n+1}}{a_n} &= \frac{2^{n+1}}{3^{2n+2}} \times \frac{3^{2n}}{2^n} && [\mathbf{M1}]\\
&= \frac{2}{3^2}\\
&= \frac{2}{9} < 1 && [\mathbf{A1}]\\
&\text{(proved).}
\end{aligned}
$$

4.
$$\sum_{n=0}^{\infty} \frac{2^{n+2}}{5^n} = \sum_{n=0}^{\infty} \frac{2^2 2^n}{5^n}$$
$$= \sum_{n=0}^{\infty} 4\left(\frac{2}{5}\right)^n \quad [\textbf{B1}]$$

Geometric series with first term 4 and common ratio $\frac{2}{5} < 1$.

$$S_\infty = \frac{4}{1-\frac{2}{5}} \qquad [\textbf{M1}]$$
$$= \frac{4}{\frac{3}{5}}$$
$$= \frac{20}{3} \qquad [\textbf{A1}]$$
$$\text{(converges).} \quad [\textbf{A1}]$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n(n+1)}$. We have

$$\frac{3}{2n(n+1)} = \frac{A}{2n} + \frac{B}{n+1} \qquad [\textbf{M1}]$$
$$= \frac{A(n+1)+B(2n)}{2n(n+1)}$$
$$= \underbrace{\frac{3}{2n}}_{a_n} - \underbrace{\frac{3}{2(n+1)}}_{a_{n+1}} \qquad [\textbf{A1}]$$
$$\text{(telescoping series)} \qquad [\textbf{A1}]$$

Let

$$S_k = \left(\frac{3}{2} - \frac{\cancel{3}}{\cancel{4}}\right) + \left(\frac{\cancel{3}}{\cancel{4}} - \frac{\cancel{3}}{\cancel{6}}\right) + \left(\frac{\cancel{3}}{\cancel{6}} - \frac{\cancel{3}}{\cancel{8}}\right) + \cdots \left(\frac{\cancel{3}}{\cancel{2k}} - \frac{3}{2(k+1)}\right)$$
$$= \frac{3}{2} - \frac{3}{2(k+1)} \qquad [\textbf{M1}]$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{3}{2n(n+1)} = \frac{3}{2} - \lim_{n\to\infty} \frac{3}{2(n+1)}$$

$$= \frac{3}{2} \qquad \textbf{[A1]}$$

(converges). **[A1]**

6.

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots + \frac{1}{n^2+1} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

(a) **Method I : Comparison Test.**

We have

$$n^2 = n^2$$

$$n^2 + 1 > n^2$$

$$\underbrace{\frac{1}{n^2+1}}_{a_n} < \underbrace{\frac{1}{n^2}}_{b_n} \qquad \textbf{[M1]}$$

where

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx$$

$$= \lim_{t\to\infty} \left. \frac{-1}{x} \right|_1^t \qquad \textbf{[M1]}$$

$$= \lim_{t\to\infty} \left(\frac{-1}{t} - (-1) \right)$$

$$= 1$$

(converges) **[A1]**

Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges by Comparison Test. **[A1]**

(b) **Method II : Integral Test.**

Let $f(x) = \dfrac{1}{x^2+1}$, where $f(x)$ is a continuous, positive and decreasing function of x.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2+1}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx && [\mathbf{M1}]\\
&= \lim_{t\to\infty} \tan^{-1} x\Big|_1^t && [\mathbf{M1}]\\
&= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 1\right)\\
&= \frac{\pi}{2} - \frac{\pi}{4}\\
&= \frac{\pi}{2} = 0.7854 && [\mathbf{A1}]
\end{aligned}$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by Integral Test. [**A1**]

7. Suppose $a_n = -\dfrac{x^n}{n}$ and $a_{n+1} = -\dfrac{x^{n+1}}{n+1}$. By using Ratio Test, let

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\
&= \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right| && [\mathbf{M1}]\\
&= \lim_{n\to\infty}\frac{n}{n+1}|x|\\
&= |x| && [\mathbf{A1}]
\end{aligned}$$

where the series converges if $\rho < 1$. This implies $|x| < 1$, or $-1 < x < 1$. [**A1**]

(a) At end-point $x = -1$,

$$\begin{aligned}\sum_{n=1}^{\infty} -\frac{(-1)^n}{n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{n}}_{b_n} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\end{aligned}$$

Since b_n are all positives, $b_{n+1} < b_n$ which is decreasing and $\lim_{n\to\infty} b_n = 0$, then by using Alternating Series Test, we found that the series converges at $x = -1$. [**M1**]

(b) At end-point $x = 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} -\frac{(1)^n}{n} &= -\sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{c_n} \\ &= -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)\end{aligned}$$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t\to\infty} \ln x \Big|_1^t \\ &= \lim_{t\to\infty} \ln t - \cancel{\ln 1} \\ &= \infty\end{aligned}$$

By using Integral Test, we found that the series diverges at $x = 1$. [**M1**]

Thus, $\sum_{n=1}^{\infty} \left(-\frac{x^n}{n}\right)$ converges on $-1 \leq x < 1$. [**A1**]