DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITI TEKNOLOGI MALAYSIA

SSH 1033 MATHEMATICAL METHODS 2 **TUTORIAL 8**

1. Evaluate the following iterated integrals.

(a) $\int_0^1 \int_{2y}^2 e^{y-x}\, dx\, dy.$ (b) $\int_1^2 \int_{x^3}^{4x^3} \frac{1}{y}\, dy\, dx.$

(c) $\int_1^9 \int_1^{y^2} \sqrt{\frac{y}{x}}\, dx\, dy.$ (d) $\int_0^\pi \int_0^{\sin x} y\, dy\, dx.$

(e) $\int_0^\pi \int_0^{2y} \sin(x+y)\, dx\, dy.$ (f) $\int_0^2 \int_{x^2}^{2x^2} x \cos y\, dy\, dx.$

(g) $\int_1^3 \int_0^1 \frac{2xy}{x^2+1}\, dx\, dy.$ (h) $\int_1^2 \int_{x^3}^{x} e^{y/x}\, dy\, dx.$

(i) $\int_0^1 \int_{y^2}^{\sqrt{y}} (\sqrt{x} - y)\, dx\, dy.$ (j) $\int_0^\pi \int_0^4 x^2 \sin^2 y\, dx\, dy.$

2. Evaluate the following double integrals.

(a) $\int_0^1 \int_{x^3}^{x^2} (x^2 - xy)\, dy\, dx.$

(b) $\int_1^{e^2} \int_1^y \ln x\, dx\, dy.$

(c) $\int_1^2 \int_0^1 \frac{y-x}{(x+y)^3}\, dx\, dy.$ **Hint:** Use $y - x = 2y - (x+y)$.

3. Evaluate the following integrals.

(a) $\iint_{\mathcal{R}} (2xy - x^2)\, dA,$ where $\mathcal{R}$ is the triangle bounded by $x = -1$, $y = -1$ and $4x + 3y = 5$.

(b) $\iint_{\mathcal{R}} (y^2 - 2x)\, dA,$ where $\mathcal{R}$ is the region bounded by $y = 1 - x$, $y = 1 + x$ and $y = 3$.

(c) $\iint_{\mathcal{R}} xy\, dA,$ where $\mathcal{R}$ is the region bounded by $y^2 = 4x$ and $2x - y = 4$.

(d) $\iint_{\mathcal{R}} y\, dA,$ where $\mathcal{R}$ is the region in the first quadrant of xy plane, bounded by the curve $y^2 = x$, the line $y = 2 - x$ and the $x-$axis.

(e) $\iint_{\mathcal{R}} (x^2 + y)\, dA,$ where $\mathcal{R}$ is the region inside a triangle with vertices $(0, 0), (1, 1)$ and $(1, -1)$.

(f) $\displaystyle\iint_{\mathcal{R}} \cos(y^2)\,dA$, where $\mathcal{R}$ is the region bounded by $2y = x$, the line $2y^2 = \pi$ and $y-$axis.

(g) $\displaystyle\iint_{\mathcal{R}} 2x\,dA$, where $\mathcal{R}$ is the region bounded by the curve $y = 1/x^2$, the lines $y = x$, $x = 2$ and $x-$axis.

(h) $\displaystyle\iint_{\mathcal{R}} (2x - y)\,dA$, where $\mathcal{R}$ is the portion of the interior of the circle $x^2 + y^2 = 9$ lying in the fourth quadrant.

(i) $\displaystyle\iint_{\mathcal{R}} x^2y^2\,dA$, where $\mathcal{R}$ is the triangle whose vertices are $(0, 0)$, $(1, 0)$ and $(1, 1)$.

(j) $\displaystyle\iint_{\mathcal{R}} e^{x^3}\,dA$, where $\mathcal{R}$ is the region bounded by $y = x^2$, the line $x = 1$, and the $x-$axis.

(k) $\displaystyle\iint_{\mathcal{R}} x\,dA$, where $\mathcal{R}$ is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

4. Evaluate the following integrals by changing the order of integration.

(a) $\displaystyle\int_0^1\int_{2y}^2 \cos(x^2)\,dx\,dy.$ (b) $\displaystyle\int_0^1\int_y^1 x^2e^{xy}\,dx\,dy.$

(c) $\displaystyle\int_0^\pi\int_x^\pi \frac{\sin y}{y}\,dy\,dx.$ (d) $\displaystyle\int_0^1\int_{x^3}^1 \frac{x^2y}{4+y^3}\,dy\,dx.$

(e) $\displaystyle\int_0^1\int_{y^2}^1 ye^{x^2}\,dx\,dy.$ (f) $\displaystyle\int_0^1\int_x^{2-x} \frac{x}{y}\,dy\,dx.$

5. Given $\displaystyle\int_0^3\int_1^{\sqrt{4-y}} (x+y)\,dx\,dy.$

(a) Sketch the region of the double integral.

(b) Interchange the order of integration. Hence evaluate the double integral.

6. Given the following integral

$$\int_{-2}^{2}\int_{-2}^{2-x^2} f(x,y)\,dy\,dx.$$

Sketch the region of integration. Obtain the equivalent integral in the reversed order.

7. Consider the following iterated integrals.

$$\int_0^1\int_x^{4\sqrt{x}} \frac{1}{1+y^2}\,dy\,dx + \int_1^4\int_x^4 \frac{1}{1+y^2}\,dy\,dx.$$

(a) Sketch the region of the integration.

(b) Write the iterated integrals as one double integral and evaluate it.

8. Use double integral to find the area of the closed region $\mathcal{R}$ described.

(a) $\mathcal{R}$ is bounded by the curves $y = \sin x$, $y = \cos x$ and the lines $x = \frac{1}{4}\pi$ and $x = \frac{1}{2}\pi$.

(b) $\mathcal{R}$ is bounded by the curve $x = -y^2$, the lines $y = x - 4$, $y = -2$, and $y = 2$.

(c) $\mathcal{R}$ is bounded by the curves $y^2 = 9x$ and $9y = x^2$.

(d) $\mathcal{R}$ is bounded by the curve $y = 5 - x^2$ and the line $y = x + 3$.

(e) $\mathcal{R}$ is bounded by the curves $y = 1/x^2$, $y = -x^2$, and the lines $x = 1$, $x = 2$.

9. Use double integrals to find the volume of the solid G described.

(a) G is the solid in the first octant bounded by $x + y + z = 4$ and the coordinate planes.

(b) G is bounded by the planes $z = 0$, $x = 1$, $x = 2$, $y = -1$, $y = 1$ and the surface $z = x^2 + y^2$.

(c) G is bounded by the cylinder $x^2 + y^2 = 9$, and the planes $z = 1$ and $x + z = 5$.

(d) G is bounded by the surfaces $z = 0$, $z = x$, and $y^2 = 2 - x$.

(e) G is bounded by the planes $z = 0$, $y = 0$, $y = x$, $x + y = 2$, and $x + x + z = 3$.

10. Show that the solid enclosed by the xy–plane and the surface $z = 4 - x^2 - y^2$ has volume

$$\iint_{\mathcal{R}} (4 - x^2 - y^2)\, dx\, dy,$$

where $\mathcal{R}$ is a region (to be specified) in the xy–plane. Hence, evaluate the volume.

11. Prove that the volume V enclosed by the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$ is given by

$$V = 2 \iint \sqrt{a^2 - x^2 - y^2}\, dx\, dy,$$

the integral being taken over the circle $x^2 + y^2 = ay$. Transform the integral into polar coordinates, and hence show that

$$V = \frac{4}{3} a^3 \left(\frac{\pi}{2} - \frac{2}{3} \right).$$

12. Use double integral to find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

13. Evaluate by changing to polar coordinates.

(a) $\displaystyle\iint_{\mathcal{R}} e^{-(x^2+y^2)} dA$, where $\mathcal{R}$ is the region above the x–axis and bounded by the semicircle $x^2 + y^2 = 1$.

(b) $\displaystyle\iint_{\mathcal{R}} x^2 y\, dA$, where $\mathcal{R}$ is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(c) $\displaystyle\iint_{\mathcal{R}} 2y\, dA$, where $\mathcal{R}$ is the smaller segment of the circle $(x-1)^2 + y^2 = 1$ cut off by the line $y = x$.

(d) $\displaystyle\iint_{\mathcal{R}} \frac{1}{1 + x^2 + y^2}\, dA$, where $\mathcal{R}$ is the region in the first quadrant bounded by the line $y = x$, the x–axis and the circle $x^2 + y^2 = 4$.

(e) $\displaystyle\iint_{\mathcal{R}} (2x - y)\, dA$, where $\mathcal{R}$ is the portion of the interior of the circle $x^2 + y^2 = 9$ lying in the fourth quadrant.

(f) $\displaystyle\iint_{\mathcal{R}} x\, dA$, where $\mathcal{R}$ is the smaller segment of the circle $x^2 + y^2 = 2y$ cut off by the line $y = x$.

14. **Evaluate the following integrals using polar coordinates.**

(a) $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$ (b) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dx.$

(c) $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} \cos\left(x^2+y^2\right)dx\,dy.$ (d) $\displaystyle\int_{-1}^0\int_{-\sqrt{1-x^2}}^0 \left(1+\sqrt{x^2+y^2}\right)dy\,dx.$

(e) $\displaystyle\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (1+x^2+y^2)\,dx\,dy.$ (f) $\displaystyle\int_0^2\int_{-\sqrt{y(2-y)}}^{\sqrt{4-y^2}} \frac{y}{x^2+y^2}\,dx\,dy.$

15. Given $\mathcal{R}$ is a region bounded by the lines $y=x$, $y=0$ and $x=1$.

(a) Sketch the region $\mathcal{R}$.

(b) Use polar coordinates to evaluate $\displaystyle\iint_{\mathcal{R}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx.$

16. Given $\mathcal{R}$ is a region on the $xy-$plane bounded by the lines $y=x$, $y=1$, $y=2$ and the $y-$axis.

(a) Sketch the region $\mathcal{R}$.

(b) Use polar coordinates to evaluate $\displaystyle\iint_{\mathcal{R}} \frac{1}{x^2+y^2}\,dA.$

17. Use polar coordinates to find the area of the region given.

(a) The region inside the circle $x^2+y^2-8y=0$ and outside the circle $x^2+y^2=9$.

(b) The region inside the circle $r=4\cos\theta$ but outside the circle $r=2$.

(c) The region inside the circle $r=3\cos\theta$ but outside the circle $r=\cos\theta$.

(d) The region inside the cardiod $r=1+\cos\theta$ but outside the circle $r=1$.

(e) The region inside the circle $r=1$ but outside the parabola $r(1+\cos\theta)=1$.

18. Find the area of the region in the first quadrant bounded by the circle $r=2\sin\theta$.

19. Find the area of the region bounded by $r=\sin 3\theta$.

20. Find the area of the region in the first quadrant bounded by the circles $(x-1)^2+y^2=1$ and $x^2+(y-1)^2=1$.

21. Evaluate $\displaystyle\iint_{\mathcal{R}} \sin\theta\,dA$,

where $\mathcal{R}$ is the region in the first quadrant bounded by the circle $r=2$ and the cardioid $r=2(1+\cos\theta)$.

22. Evaluate the area in the first quadrant bounded by the curves $r=2$, $r=4\sin\theta$ and $r=4\cos\theta$.

23. Find the volume of the solid which is

(a) bounded by the surface $r^2+z^2=4$.

(b) bounded by the surfaces $z=0$, $2z=x^2+y^2$, and $x^2+y^2=4$.

(c) bounded by the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.
(d) bounded by the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 - 2y = 0$.
(e) bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 2x$.

24. Consider the region $\mathcal{R}$ in the $xy-$plane bounded by the lines $y = 0$, $x + y = 2$ and the curve $y = x^2$.

(a) Show that the area of the region $\mathcal{R}$ is $\frac{5}{6}$.
(b) Find the moment of inertia of $\mathcal{R}$ about the $z-$axis (or with respect to the origin), assuming the density σ is constant.

25. Find the centroid of the lamina with uniform density $\delta(x, y) = 1$ bounded by the parabola $y = 2 - 3x^2$ and the line $3x + 2y = 1$.

26. Find the centre of gravity of the lamina with density $\delta(x, y) = x^2 + y^2$, satisfying the inequalities $x^2 + y^2 \leq 9$ and $y \geq 0$.

27. A lamina on the $xy-$plane has the shape of a semicircle $x^2 + y^2 \leq 4$, $y \geq 0$. Find the centre of mass if the density at each point is proportional to its distance from the origin.

28. Find the moment of inertia I_x of the lamina which is bounded by the graph $y = 1 - x^2$ and $x-$axis and has density $\delta(x, y) = x^2$.

29. Find the mass of a lamina that is bounded by the curve

$$(x - 1)^2 + y^2 = 1 \quad \text{and} \quad \text{the line} \quad y = x$$

if the density function of the lamina is $\rho(x, y) = 3\text{kgm}^{-2}$.

30. Find the mass of a lamina that is bounded by the curves

$$x = y^2 \quad \text{and} \quad x = y$$

if the density function of the lamina is $\sigma(x, y) = 2(y + 1)\text{kgm}^{-2}$. Hence find its moment about the $x-$axis and evaluate $\bar{y}$, the $y-$coordinates of the center of mass of the lamina.

31. A lamina is bounded by $y = x^2 + 4$ and $y = 8 - x^2$ with the density function $\sigma(x, y) = \frac{1}{y^2}$. Find the moment of inertia of the lamina about the $x-$axis.

32. Find the mass, center of mass, and the moment of inertia of a lamina that is bounded by the curve $x = 1 - y^2$ and the coordinates axes in the first quadrant with density function $\delta(x, y) = y\text{kgm}^{-2}$.

33. Find the mass and the center of mass of a lamina that is bounded by the circle $x^2 + y^2 = 2x$ and the coordinates axes in the first quadrant with density function $\delta(x, y) = xy^2\text{kgm}^{-2}$.

34. Find the mass of a lamina bounded by the lines $y = 0, y = 3, x = 1$, and the curve $x = \sqrt{4 - y}$, if the density at the point (x, y) is $\delta(x, y) = x + y$.

35. Find the center of mass of a lamina bounded by $y = x^2$ and $y = 4$ with density $\delta(x, y) = y$.

36. Find the moment of inertia about the $x-$axis, I_x, of a lamina bounded by a parabola $x = y^2$, the line $y = x - 2$ and the $x-$axis in the first quadrant with density $\sigma(x, y) = 1$.

37. Using double integrals in polar coordinates, find the mass of a semicircular lamina $x^2 + y^2 = 4$, with $y \geq 0$, and density function given by $\delta(x, y) = \sqrt{4 - x^2 - y^2}$.