Chapter 1

Continuous Dynamical Systems (CDS)

Chapter 2

Periodic Orbits

2.1 limit cycles

2.2 Bifurcation

Chapter 3

Discrete Dynamical Systems(DDS)

3.1 Introduction

In this chapter we will examine discrete dynamical systems that are governed by difference equation of the form

$$x_{n+1} = f(x_n), \quad x_0 \quad \text{specified.} \tag{3.1}$$

We will develop techniques for analysing nonlinear difference equations and explaining some of the naturally arising phenomena such as bifurcation, chaos and fractals. The emphasis will be on one-dimensional discrete dynamical systems, and therefore the function f in (3.1) will usually be a real-valued function of a real variable. However, we shall also consider more abstract cases in order to introduce some of the notation, terminology and concepts associated with discrete dynamical systems and iterated maps.

For a DDS, we suppose that the evolution through time of a particular system occurs in discrete steps, e.g. in steps of size $\triangle t$. If we write $\phi(x, n)$ to denote the value at time $t = n\triangle t$ of the system that took the value x at time $t = 0$, then for one-dimensional DDS, ϕ is defined on $\mathbb{R} \times \mathbb{N}$. Any such function ϕ satisfying

1. $\phi(x, 0) = x, \quad \forall x \in \mathbb{R}$
2. $\phi(\phi(x, n), m) = \phi(x, n + m) \quad \forall x \in \mathbb{R}, \quad \forall n, m \in \mathbb{N}$

DDS example
As an example of how a one-dimensional DDS might be generated, consider the function (or map) $f : \mathbb{R} \to \mathbb{R}$ which satisfies the first-order difference equation (or iteration)

$$x_{n+1} = f(x_n), \quad x_0 \text{ specified.}$$

For $n \in \mathbb{N}$, we define the nth **iterate** or n-**fold composition** of f to be

$$f^n = f \circ f \circ f \circ f \cdots \circ f \quad (n \text{ terms}).$$

Note that f^n does not mean "f to the power of n" here, but n application of f

$$f^2(x) = f(f(x)), \qquad f^3(x) = f\left(f^2(x)\right) = f(f(f(x))), \qquad \text{etc}$$

If we also define f^0 by $f^0(x) = x \ \forall x \in \mathbb{R}$, it is then follows that

1. $f^0(x) = x \quad \forall x \in \mathbb{R}$
2. $f^n\left(f^m(x)\right) = f^{n+m}(x) \quad \forall x \in \mathbb{R}, \quad \forall n, m \in \mathbb{N}.$

Writing

$$\phi(x, n) = f^n(x) \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}$$

we see that $\phi(x, n)$, satisfies the properties of a a discrete dynamical system.

When faced with equation (3.1), the main objectives are namely

1. Given an initial value x_0, determine the **asymptotic** (long term) behaviour of x_n (i.e what happens to x_n as $n \to \infty$).
2. Identify initial values which give rise to sequences having the same asymptotic behaviour.
3. Examine the **stability** of solutions, i.e determine whether a small change to the initial value x_0 leads to only a small change in each x_n, $n = 1, 2, \cdots$

3.2 Metric Spaces

Some of the terminology and concepts that are associated with discrete dynamical system generated by equation (3.1) will be introduced.

Definition 1 *A* **metric space** *consists of a non-empty set X together with a* **metric** $d : X \times X \to \mathbb{R}$ *such that*

1. $d(x, y) \geq 0 \ \forall x, y \in X$ *and* $d(x, y) = 0 \iff x = y$ *in* X*.*

2. $d(x, y) = d(y, x) \ \forall x, y \in X$*.*

3. $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$ *(the* **triangle inequality***)*

Definition 2 (Convergence of Sequence) *Let $\{x_n\} \subset X$ where X is a metric space with metric d*

1. $x_n \to x$ *in* X *as* $n \to \infty$ *if, for any given* $\epsilon > 0$, $\exists N \in \mathbb{N}$ *such that*

$$d(x_n, x) < \epsilon \ \ \forall n \geq N.$$

2. $\{x_n\}$ *is a* **Cauchy sequence** *in* X*, if for any given* $\epsilon > 0$, $\exists N \in \mathbb{N}$*, such that*

$$d(x_n, x_m) < \epsilon \ \ \forall n, m \geq N.$$

3. X *is a* **complete** *metric space if every cauchy sequence in X is convergent.*

Definition 3 (Continuity) *Let X, Y be metric spaces with metric d_1, d_2 respectively and let $f : X \to Y$ be a function.*

1. f *is* **continuous** *at* $x_0 \in X$ *if, for any given* $\epsilon > 0$, $\exists \delta > 0$ *such that*

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \epsilon.$$

2. f *is* **continuous** *if f is continuous at each point in X*

3. f *is a* **homeomorphism** *if f is $1{-}1$, continuous, onto and has continuous inverse* $f^{-1} : Y \to X$*.*

Definition 4 *Let $G \subset X$ where X is a metric space with metric d. Then G is said to be* **dense** *in X if, for any given $x \in X$ and $\epsilon > 0$, $\exists y \in G$ such that $d(x, y) < \epsilon$. Equivalently, G is dense in X if, for any given $x \in X$, $\exists \{x_n\} \subset G$ such that $x_n \to x$ in X as $n \to \infty$. We write $\bar{G} = X$ and $\bar{G}$ the* **closure** *of G.*

We introduce a specific metric space and mapping which we will use later to illustrate some ideas. This is the metric space comprising the set

$$\Sigma_2 = \{s = \{s_k\}_{k=0}^{\infty}\,, \;\; s_k \in \{0,1\}\,, \;\; k = 0,1,2,\cdots\}$$

(i.e the set of all infinitely long sequences comprising ones and zeros) and the metric d defined on $\Sigma_2 \times \Sigma_2$ by

$$d(s,t) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k}.$$

Lemma 1 *Let $\{s_k\}$, $\{t_k\} \in \Sigma_2$. Then*

1. *If $s_k = t_k$ for $k = 0,1,2,\cdots,n$ then $d(s,t) \le \frac{1}{2^n}$.*
2. *If $d(s,t) < \frac{1}{2^n}$, then $s_k = t_k$ for $k = 0,1,2,\cdots,n$.*

Proof

1. If $s_k = t_k$ for $k = 0,1,2,\cdots,n$ then

$$\begin{aligned} d(s,t) &= \sum_{k=0}^{n} \frac{|s_k - s_k|}{2^k} + \sum_{k=n+1}^{\infty} \frac{|s_k - t_k|}{2^k} \\ &\le \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^n} \end{aligned}$$

2. Let $d(s,t) < \frac{1}{2^n}$ but suppose $s_i \ne t_i$ for some $i \in \{0,1,2,\cdots,n\}$. Then

$$d(s,t) \ge \frac{1}{2^i} \ge \frac{1}{2^n}.$$

This is a contradiction and so $s_k = t_k$ for $k = 0,1,2,\cdots,n$.

Theorem 1 (Shift map on Σ_2) *Let $\sigma : \Sigma_2 \to \Sigma_2$ be defined by*

$$\sigma\left(\{s_k\}_{k=0}^{\infty}\right) = \{s_{k+1}\}_{k=0}^{\infty}\,,$$

that is,

$$\sigma\left(\{s_0, s_1, s_2, \cdots\}\right) = \{s_1, s_2, \cdots\}\,.$$

Then σ is continuous on Σ_2.

Proof Given $s, t \in \Sigma_2$ and $\epsilon > 0$, we must show that $\exists\, \delta > 0$ such that

$$d(s,t) < \delta \Rightarrow d(\sigma(s), \sigma(t)) < \epsilon.$$

For a given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $0 < \frac{1}{2^N} < \epsilon$ (this can be done as $\frac{1}{2^n} \to 0$ as $n \to \infty$) and set $\delta = \frac{1}{2^{N+1}}$. It then follows that

$$\begin{aligned} d(s,t) < \delta &\Rightarrow d(s,t) < \frac{1}{2^{N+1}} \\ &\Rightarrow s_k = t_k \text{ for } k = 0,1,2,\cdots,N+1 \quad (\text{ Lemma 1(2)}) \\ &\Rightarrow s_{k+1} = t_{k+1} \text{ for } k = 0,1,2,\cdots,N \\ &\Rightarrow d(\sigma(s),\sigma(t)) \le \frac{1}{2^N} \\ &\Rightarrow d(\sigma(s),\sigma(t)) < \epsilon. \end{aligned}$$

Examples 3A

1. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be in $\mathbb{R}^n$. Show that the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}$$

is a metric on $\mathbb{R}^n$.

solution:
We have

(a) $d(\mathbf{x},\mathbf{y}) \ge 0 \quad d(\mathbf{x},\mathbf{y})$ and $d(\mathbf{x},\mathbf{y}) = 0 \Longleftrightarrow \mathbf{x} = \mathbf{y}$ in $\mathbb{R}^n$;

(b) $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x}) \quad \forall\ \mathbf{x},\mathbf{y} \in \mathbb{R}^n$;

(c) $d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} ((x_i - z_i) + (z_i - y_i))^2\right]^{\frac{1}{2}}$
$\le \left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} (z_i - y_i)^2\right]^{\frac{1}{2}}$ (by the **triangle inequality**)
$= d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y}) \quad \forall\ \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$

2. Show that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$

solution:
From standard results on real numbers, given any $x \in \mathbb{R}$ and $\epsilon > 0$, there exists a rational number $y = \dfrac{p}{q}$ such that $y \in (x - \epsilon, x + \epsilon)$ (so that $d(x,y) < \epsilon$) and therefore the set $\mathbb{Q}$ is dense in $\mathbb{R}$.

3. Let S^1 denote the unit circle in the plane, i.e. $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Show that the function on S^1 defined by

$$d(P, Q) = \text{ length of arc connecting } P \text{ to } Q, \qquad P, Q \in S^1$$

is a metric.

solution:

(a) $d(P, Q) \geq 0 \quad \forall P, Q \in S^1$ and $d(P, Q) = 0 \Longleftrightarrow P = Q$ in S^1

(b) $d(P, Q) = d(Q, P) \quad \forall P, Q \in S^1$

(c) $d(P, Q) \leq d(P, R) + d(R, Q) \quad \forall P, Q, R \in S^1$

4. Let $\Sigma_2 = \{s = \{s_k\}_{k=0}^{\infty}\,,\ \text{where} \quad s_k \in \{0, 1\}\,,\ \text{for} \quad k = 0, 1, 2, \cdots\}$ and define d on $\Sigma_2 \times \Sigma_2$ by

$$d(s, t) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k}, \quad s = \{s_k\}\,, t = \{t_k\} \in \Sigma_2.$$

Show that d is a metric on Σ_2.

solution:

3.3 Iterated Maps: General Definitions and Results

Let X be a metric space with metric d and let $f : X \to X$ be a continuous function. We are interested in the dynamical system generated by the difference equation (3.1). If we let $x_0 = x \in X$ and again denote the n^{th} iterate of f by f^n then we can write

$$x_n = f^n(x) \ \ n = 0, 1, 2, \cdots$$

where f^0 is defined by $f^0(x) = x$. Functions f which generate dynamical system via equation of the form (3.1) are usually called **mappings** or **maps**. In this chapter we will present some key definitions for properties of iterated maps.

3.3.1 Orbits and Periodicity

We begin by introducing the idea of **orbits**.

Definition 5 (Orbits) *Let $x \in X$, and let $f : X \to X$.*

1. $\gamma_+(x) = \{f^n(x) : n = 0, 1, 2, \cdots\}$ *is called the* **positive** *(or* **forward***)* **semi-orbit** *of x under f*

2. *If f is a* **homeomorphism** *then*

$$\gamma_-(x) = \left\{f^{-n}(x) : n = 0, 1, 2, \cdots\right\} \ \ \left(f^{-n} \equiv n^{th} \text{iterate of } f^{-1}\right)$$

   is called the **negative** *(or* **backward***)* **semi-orbit** *of x under f.*

3. $\gamma(x) = \{f^n(x) : n \in \mathbb{Z}\} = \gamma_-(x) \cup \gamma_+(x)$ *is called the* **full orbit** *of x under f.*

Remarks

1. We will follow the convention that only **distinct** points from the sequence $x, f(x), f^2(x), \cdots$ are included in $\gamma_+(x)$ (and similarly for $\gamma_-(x)$ and $\gamma(x)$).

2. If we define $\phi(x, n) \equiv f^n(x)$ then

$$\gamma_+(x) = \{\phi(x, n) : n \in \mathbb{N}\}.$$

   Similar expression can be obtained for the negative semi-orbit and full orbit.

One type of orbits which are particularly simple are **periodic orbits**.

Definition 6 (Fixed and Periodic Points) *Let $f : X \to X$.*

1. *$p \in X$ is a* **fixed point** *(***equilibrium point***) for f if $f(p) = p$. In this case*

$$f^n(p) = p \ \ \forall n = 0, 1, 2, \cdots$$

   and so $\gamma_+(p) = \{p\}$.

2. *The set of fixed points for f is denoted by*

$$\mathrm{Fix}(f) = \{p \in X : f(p) = p\}.$$

3. *If p_1 and p_2 are such that*

$$f(p_1) = p_2, \quad f(p_2) = p_1,$$

   then the points $p_1, p_2 \in X$ form a **period 2-cycle** *for f. We call p_1 and p_2* **periodic points of period 2** *for f. Note that*

$$f^2(p_1) = p_1 \ \ and \ \ f^2(p_2) = p_2$$

   so $p_1, p_2 \in \mathrm{Fix}\left(f^2\right)$.
   Also, $\gamma_+(p_1) = \gamma_+(p_2) = \{p_1, p_2\}$. This is a **periodic orbits of period 2**.

4. *More generally, $p \in X$ is called a* **periodic point of period n** *if $f^n(p) = p$. In addition, it has* **prime period** *n if $f^k(p) \neq p$ for any $k = 1, 2, \cdots, n-1$. In such a case*

$$\gamma_+(p) = \left\{p, f(p), f^2(p), \cdots, f^{n-1}(p)\right\}$$

   is called a **periodic orbit of prime period n** *or, more simply, an* **n-cycle**.

5. *The set of all periodic points of (not necessarily prime) period n is denoted by*

$$\mathrm{Per}_n(f) = \{p \in X : f^n(p) = p\}$$

   and

$$\mathrm{Per}(f) = \bigcup_{n=1}^{\infty} \mathrm{Per}_n(f).$$

Remarks

1. If $p \in \mathrm{Per}_n(f)$ then $f(p), f^2(p), \cdots, f^{n-1}(p)$ are also in $\mathrm{Per}_n(f)$.

2. $\mathrm{Per}_n(f) = \mathrm{Fix}(f^n)$.

3. $p \in \mathrm{Per}_n(f) \Rightarrow p \in \mathrm{Per}_{kn}(f) \quad \forall k = 1, 2, \cdots$
   In particular, $\mathrm{Fix}(f) \subseteq \mathrm{Per}_n(f) \quad \forall n \in \mathbb{N}$

In practice, fixed points can be found **algebraically** (by solving the equation $f(x) = x$)) or **graphically** (by finding any intersections of the graph $y = f(x)$ with a straight line $y = x$)

Examples 3B

1. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 1$. Find the fixed points of $f(x)$ and $f^2(x)$ and write down $\gamma_+(\sqrt{2})$ under f.

   Solution: To find the fixed points

$$\begin{aligned} f(x) &= x \\ x^2 - 1 &= x \\ x^2 - x - 1 &= 0 \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

   So $\mathrm{Per}_1(f) = \mathrm{Fix}(f) = \left\{ \frac{1 \pm \sqrt{5}}{2} \right\}$

   Fixed point for $f^2(x)$

$$\begin{aligned} f^2(x) &= x \\ f(f(x)) &= f(x^2 - 1) = x \\ (x^2 - 1)^2 - 1 &= x \\ x^4 - 2x^2 - x &= 0 \\ (x^2 - x - 1)(x^2 + x) &= 0 \\ x &= 0, -1, \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

   $x^2 - x - 1$ must be a factor since two solutions of $f^2(x) = x$ are $\frac{1 \pm \sqrt{5}}{2}$.

   Note that $0, -1$ have prime period 2 and form a 2-cyle ($f(0) = -1, f(-1) = 0$).

   For $\gamma_+(\sqrt{2})$ under f, we need the sequence $\left\{ f^0(\sqrt{2}), f^1(\sqrt{2}), f^2(\sqrt{2}), \cdots) \right\}$. As $f(x) = x^2 - 1$, this gives $\gamma_+(\sqrt{2}) = \left\{ \sqrt{2}, 1, 0, -1, 0, -1, \cdots \right\}$
   refer to maple ex3Bno1 for analysis using graph

2. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$. Find $\mathrm{Per}(f)$.

   Solution:

   $f(x) = x \Leftrightarrow x^3 = x \Leftrightarrow x^3 - x = 0 \Leftrightarrow x(x^2 - 1) = 0 \Leftrightarrow x = 0, \pm 1$
   So $\mathrm{Per}_1(f) = \mathrm{Fix}(f) = \{0, \pm 1\}$.

In general

$$f^n(x) = x \Leftrightarrow x^{3n} = x \Leftrightarrow x^{3^n} - x = 0 \Leftrightarrow x\left(x^{3^n-1} - 1\right) = 0$$

Since $3^n - 1$ is even, $x = 0, \pm 1$.
So $\text{Per}_n(f) = \{0, \pm 1\} = \text{Fix}(f)\ \forall n$ and $\text{Per}(f) = \text{Fix}(f)$,

3. Let $f_\mu : \mathbb{R} \to \mathbb{R}$, $f_\mu(x) = \mu x(1-x)$, $\mu > 0$. This is known as the **logistic function**. Find $\text{Per}_1(f_\mu)$ and $\text{Per}_2(f_\mu)$.

   Solution:

$$\begin{aligned} f_\mu(x) &= x \\ \mu x(1-x) &= x \\ x(\mu - \mu x - 1) &= 0 \Rightarrow x = 0 \text{ or } x = \frac{\mu - 1}{\mu} \end{aligned}$$

so

$$\text{Per}_1(f_\mu) = \text{Fix}(f_\mu) = \begin{cases} \left\{0, \frac{\mu-1}{\mu}\right\} & \mu \neq 1 \\ \{0\} & \mu = 1 \end{cases}$$

$$\begin{aligned} f_\mu^2(x) &= x \\ f_\mu(\mu x(1-x)) &= x \\ \mu[\mu x(1-x)](1 - \mu x(1-x)) &= x \\ \mu^2 x(1-x)(1 - \mu x(1-x)) - x &= 0 \\ \mu^3 x^4 - 2\mu^3 x^3 + \mu^3 x^2 + \mu^2 x^2 - \mu^2 x + x &= 0 \\ x(\mu - \mu x - 1)\left(\mu^2 x^2 - \mu^2 x - \mu x + \mu + 1\right) &= 0 \end{aligned}$$

$x(\mu - \mu x - 1)$ must be a factor since two solutions of $f_\mu^2(x) = x$ are $\text{Per}_1(f_\mu)$.

$$x = 0, \frac{\mu - 1}{\mu} \qquad \text{or} \qquad \mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1 = 0.$$

The second equation gives

$$x = \frac{\mu^2 + \mu \pm \sqrt{(\mu^2 + \mu)^2 - 4\mu^2(\mu + 1)}}{2\mu^2} = \frac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu} \equiv q_\mu^+, q_\mu^-$$

Thus f_μ has a 2-cycle $\left\{q_\mu^+, q_\mu^-\right\}$ if $\mu^2 - 2\mu - 3 = (\mu - 3)(\mu + 1) > 0$, that is, if $\mu < -1$ or $\mu > 3$. Since we assume $\mu > 0$, so if $\mu > 3$, f_μ has a 2-cycle.

[see maple ex3BFP]

3.3.2 Stability and ω-Limit Sets

Definition 7 (Attracting and Repelling Fixed points) *Let $f : X \to X$ where X is a metric space with metric d.*

1. *Let $p \in \mathrm{Fix}(f)$. Then*
   (a) *p is an* **attracting** *(or* **locally asymptotically stable***) fixed point if $\exists \epsilon > 0$ such that*
   $d(x,p) < \epsilon \Rightarrow f^k(x) \to p$ *as* $k \to \infty$.
   (b) *p is a* **repelling** *(or* **unstable***) fixed point if $\exists \epsilon > 0$ such that*
   $0 < d(x,p) < \epsilon \Rightarrow d\left(f^k(x), p\right) > \epsilon$ *for some (but not necessarily all) values of k*

2. *The 2-cycle p_1, p_2 is an* **attracting** *2-cycle for f if $\exists \epsilon_1, \epsilon_2 > 0$ such that*

$$\left.\begin{array}{ll} d(x,p_1) < \epsilon_1 & \Rightarrow f^{2k}(x) \to p_1, f^{2k+1}(x) \to p_2 \\ d(x,p_2) < \epsilon_2 & \Rightarrow f^{2k}(x) \to p_2, f^{2k+1}(x) \to p_1 \end{array}\right\} \text{ as } k \to \infty$$

   Attracting n-cycles ($n > 2$) can be defined similarly.

3. *A set $S \subset X$ is said to be*
   (a) **positively invariant** *under f if $f(S) \subseteq S$.*
   (b) **negatively invariant** *under f if $S \subseteq f(S)$.*
   (c) **invariant** *under f if $S = f(S)$.* **Note:** $f(x) = \{y = f(x) : x \in S\}$

4. *A set $S \in X$ is an* **attracting set** *(or* **attractor***) for f if*
   (a) *S is invariant (i.e $f(S) = S$)*
   (b) *$\exists \epsilon > 0$ such that*

$$\mathrm{dis}(x,S) < \epsilon \Rightarrow \mathrm{dist}\left(f^k(x), S\right) \to 0 \text{ as } k \to \infty,$$

   where $\mathrm{dist}(x,S) = \inf\{d(x,y) : y \in S\}$

Remarks

1. Not all fixed points can be categorized as either attracting or repelling; e.g some are **weakly attracting** (or **semi-stable**): this will be discussed further later.

2. By definition, if p_1, p_2 form an attracting 2-cycle for f, then p_1 and p_2 are both attracting fixed points for f^2. When the 2-cycle consists of repelling fixed points for f^2, we say that p_1 and p_2 form a **repelling** or **unstable 2-cycle**. Similar comments apply to n-cycles ($n > 2$) i.e. the n-cycle $\{p_1, p_2, p_3, \cdots, p_n\}$ is attracting (repelling) if each p_i is an attracting (repelling) fixed point of f^n.

3. Suppose that S is invariant under f and let $\phi(x,n) \equiv f^n(x)$. Then it follows immediately that
$$\phi(S,n) = S \quad \forall n \in \mathbb{N}.$$

4. Expressed more simply, a set $S \subset X$ is an attractor if $f^n(x) \to S$ as $n \to \infty$ for all pint x "sufficiently close" to S, where $f^n(x) \to S$ as $n \to \infty$ means that given $\delta > 0$, $\exists N$ such that each $f^n(x)$ is within δ of some point y_n in S $\forall n \geq N$. In addition, $x \in S \Rightarrow \gamma_+(x) \subset S$. Simple examples of attractors are
   (a) $S = \{p\}$ where p is an attracting fixed point.
   (b) $S = \{p_1, p_2\}$ where p_1, p_2 form an attracting 2-cycle.
   (c) $S = \{p_1, p_2, \cdots, p_n\}$ where $p_1, p_2, \cdots, p_n$ form an attracting n-cycle.

Definition 8 (Stability and ω-limit Sets) *Let $f : X \to X$, where X is a metric space with metric d.*

1. *$x \in X$ is* **forward asymptotic** *to $p \in \mathrm{Fix}(f)$ if $f^k(x) \to p$ as $k \to \infty$. The* **stable set** *of p is defined by*
$$W^s(p) = \left\{x \in X : f^k(x) \to p \ as \ k \to \infty\right\}.$$
Similarly, $x \in X$ is **forward asymptotic** *to $p \in \mathrm{Per}_n(f)$ if*
$$f^{nk}(x) = (f^n)^k(x) \to p \ as \ k \to \infty,$$
and **stable set** *of p is defined by*
$$W^s(p) = \left\{x \in X : f^{nk}(x) \to p \ as \ k \to \infty\right\}.$$

2. *If $f^{-1}(x)$ exists, we say that*
   (a) $x \in X$ is **backward asymptotic** *to $p \in \mathrm{Fix}(f)$ if*
$$f^{-k}(x) = \left(f^{-1}\right)^k(x) \to p \ as \ k \to \infty$$
   (b) $x \in X$ is **backward asymptotic** *to $p \in \mathrm{Per}_n(f)$ if*
$$f^{-nk}(x) \to p \ as \ k \to \infty.$$
The **unstable sets** *of $p \in \mathrm{Fix}(x)$ and $p \in \mathrm{Per}_n(f)$ are defined by*
$$\begin{aligned} W^u(p) &= \left\{x \in X : f^{-k}(x) \to p \ as \ k \to \infty\right\} \\ W^u(p) &= \left\{x \in X : f^{-nk}(x) \to p \ as \ k \to \infty\right\} \end{aligned}$$
respectively

3. *$y \in X$ is an ω-**limit points** of $x \in X$ if there exists a subsequence $\{f^{n_r}(x)\}$ of $\{f^n(x)\}$ such that $f^{n_r}(x) \to y$ as $n_r \to \infty$. The ω-**limit set** $\omega(x)$ of x is the set of all ω-limit points of x. If f^{-1} exists, then we can also define α-limit points and the α-limit set in an analogous manner simply by replacing f by f^{-1}. Note that, in terms of discrete dynamical systems*

$$\phi(x,n) \equiv f^n(x)$$

Theorem 2 (Results on ω-limit sets) *Let $f : X \to X$ where X is a metric space with metric d.*

1. *$\omega(x)$ is positively invariant under f for each $x \in X$ i.e. $f(\omega(x)) \subseteq \omega(x)$. In some cases it can be shown that $\omega(x)$ is invariant under f. i.e $f(\omega(x)) = \omega(x)$, e.g. when $X = \mathbb{R}^n$ and $\gamma_+(x)$ is bounded.*
2. *Let $p \in \mathrm{Fix}(f)$. Then*
   (a) $\omega(p) = \{p\}$
   (b) $\omega(x) = \{p\} \quad \forall x \in W^s(p)$
3. *Let $S = \{p_1, p_2, \cdots, p_k\}$ be a k-cycle for f. Then*
   (a) $\omega(p_1) = \omega(p_2) = \cdots = \omega(p_k) = S$
   (b) *$\omega(x) = S$ whenever $x \in W^s(p_i)$ for some $p_i \in S$*

Proof

1.

$$\begin{aligned}
y \in \omega(x) \quad &\Rightarrow \quad f^{n_r}(x) \to y \text{ as } n_r \to \infty \text{ for some sequence } \{f^{n_r}(x)\} \\
&\Rightarrow \quad f\left(f^{n_r}(x)\right) \to f(y) \text{ as } n_r \to \infty \text{ since } f \text{ is continuous} \\
&\Rightarrow \quad f^{n_r+1}(x) \to f(y) \text{ as } n_r + 1 \to \infty \\
&\Rightarrow \quad f(y) \in \omega(x).
\end{aligned}$$

2. (a) $p \in \mathrm{Fix}(f) \Rightarrow f^n(p) = p \; \forall n = 0, 1, 2, \cdots \Rightarrow \omega(p) = \{p\}$
   (b)

$$\begin{aligned}
x \in W^s(p) \quad &\Rightarrow \quad f^n(x) \to p \text{ as } n \to \infty \\
&\Rightarrow \quad \text{every subsequence of } \{f^n(x)\} \text{ also converges to } p \\
&\Rightarrow \quad \omega(x) = \{p\}\,.
\end{aligned}$$

3. (a) Let $S = \{p_1, p_2, \cdots, p_k\}$ be a k-cycle for f. Then

$$\{f^n(p_i)\}_{n=0}^{\infty} = \{p_i, p_{i+1}, \cdots, p_k, p_1, \cdots, p_{i-1} | p_i, p_{i+1}, \cdots, p_{i-1} | \cdots\}$$

and so the only possible convergent subsequences are those that ultimately are constant. i.e. of the form

$$\{\text{ finitely many terms } p_j, p_j, p_j, \cdots\}$$

for some fixed $p_j \in S$. Hence $\omega(p_i) = S$.

(b) $x \in W^s(p_i) \Rightarrow f^{nk}(x) \to p_i, f^{nk+1}(x) \to p_{i+1}, \cdots, f^{nk+k-1}(x) \to p_{i-1}$.
Consequently the only possible convergent subsequences of $\{f^n(x)_{n=0}^{\infty}\}$ are of the form

$$\left\{\text{finitely many terms, infinitely many terms from } \left\{f^{nk+j}(x)\right\} \text{(for fome fixed } j)\right\}$$

and so $\omega(x) = S$.

Definition 9 (Aperiodicity) *If $\omega(x)$ contains* **infinitely many** *points then $\gamma_+(x)$ is said to be* **aperiodic**.

Together, these concepts can be used to analyze the behaviour of difference equations, either analytically or using graphs. One important idea for the latter approach is a so-called **cobweb diagram**, which is constructed as follows: given $x_{n+1} = f(x_n)$ and an initial condition x_0, draw a vertical line from x_0 until it intersects the graph of f (the height is output x_1). This could be repeated to get x_2, but it is more convenient to draw a horizontal line until it intersects the diagonal line $x_{n+1} = x_n$, and then move vertically to the curve again. Repeat the process n times to obtain the first n points of the orbit.

Examples 3C

3.3.3 The Banach Contraction Mapping Principle

To end this chapter, we quote an important theorem which guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces.

Theorem 3 (Banach Contraction Mapping Principle) *Let X be a complete metric space with metric d and let $f : X \to X$ be a function with the property that*

$$d(f(x), f(y)) \leq \alpha d(x, y), \quad \forall x, y \in X$$

for some constant $\alpha < 1$ (f is said to be a **contraction***). Then f has exactly one fixed point $p \in X$. Also $W^s(p) = X$ since $f^n(x) \to p$ as $n \to \infty$ $\forall x \in X$.*

Corollary 1 *If $f : X \to X$ is a contraction with fixed point p, then* $\mathrm{Per}(f) = \{p\}$.

Corollary 2 *Let* $f \in C^1[a,b]$ *that is,* f *and* f' *are both continuous on* $[a,b]$ *with* $f([a,b]) \subseteq [a,b]$ *and* $|f'(x)| < 1\ \forall x \in [a,b]$*. Then* f *has exactly one fixed point* $p \in [a,b]$*. Moreover* $\mathrm{Per}(f) = p$ *and* $W^s(p) = [a,b]$

Proof
First note that $[a,b]$ equipped with the metric $d(x,y) = |x-y|$, is a complete metric space. Since f' is continuous on $[a,b]$ and $|f'(x) < 1|\ \forall x \in [a,b]$, there exists $\alpha < 1$ such that $|f'(x)| \leq \alpha\ \forall x \in [a,b]$. Moreover, by the Mean Value Theorem (MVT)

$$\begin{aligned} |f(y) - f(x)| &= |f'(c)||y-x|, \text{ for some } c \text{ between } x \text{ and } y, \\ &\leq \alpha|y-x| \quad \forall x,y \in [a,b] \end{aligned}$$

Hence f is a contraction on $[a,b]$ and so the result follows from the contraction mapping principle.

Examples 3D Let $I = [0,1]$ and consider the logistic map $f_\mu(x) = \mu x(1-x)$, $x \in I$; $\mu > 0$. Show that $f_\mu : I \to I$ if $0 < \mu \leq 4$. Find any values of μ for which f_μ is a contraction and identify any fixed points. What deduction can you make about the asymptotic behaviour of f_μ in this case?

Solution: Since

$$0 \leq f_\mu(x) \leq f_\mu\left(\frac{1}{2}\right) = \frac{\mu}{4} \quad \forall x \in I,$$

it follows that $f_\mu : I \to I$ if $0 < \mu \leq 4$. Also $|f'_\mu(x)| = |\mu - 2\mu x| = \mu|1-2x| \leq \mu\ \forall x \in I$. From example 3B, we know that the fixed points are

$$\mathrm{Fix}(f_\mu) = \left\{0, \frac{\mu-1}{\mu}\right\} \quad \text{so } f_\mu \text{ has } \left\{\begin{array}{ll} 2 & \text{fixed points in } I \text{ if } \mu > 1 \\ 1 & \text{fixed point in } I \text{ if } 0 < \mu \leq 1 \end{array}\right.$$

Hence for $0 < \mu < 1$ we can state that $f_\mu^n(x) \to 0$ as $n \to \infty\ \forall x \in I$ (by corollary 2). Furthermore, no $x \in (0,1]$ is in $\mathrm{Per}_k(f_\mu)$ for any k. Because if $x \in [0,1]$ is in $\mathrm{Per}_k(f_\mu)$, then $f_\mu^n(x)$ will NOT go to 0 and $n \to \infty$, which gives a contradiction.