Examples 3C

1. **Linear map**. Let $f_a : \mathbb{R} \to \mathbb{R}, f_a(x) = ax, a \neq 0$, with inverse $f_a^{-1}(x) = \dfrac{x}{a}$. Analyse the behaviour of the associated difference equation $x_{n+1} = ax_n$.

   Solution:
   Fixed points: $f_a(x) = x \Rightarrow ax = x$ so

   $$\mathrm{Per}_1(f_a) = \mathrm{Fix}(f_a) = \begin{cases} \{0\} & if \quad a \neq 1 \\ \mathbb{R} & if \quad a = 1 \end{cases}$$

   We now consider the asymptotic behaviour of

   $$f_a^n(x) = a^n x, \qquad f_a^{-n} = \frac{x}{a^n}, \quad \forall n \in \mathbb{Z}$$

   for different values of a.

   (a) $|a| < 1$: We have
      i. $f_a^n(x) = a^n x \to 0$ as $n \to \infty$ $\forall x \in \mathbb{R}$
      ii. $|f_a^{-n}(x)| \to \infty$ as $n \to \infty$ $\forall x \neq 0$

      so 0 is an attracting fixed point with $W^s(0) = \mathbb{R}$ and $W^u(0) = \{0\}$. Hence $\omega(x) = \{0\}$ $\forall x \in \mathbb{R}$.

   (b) $|a| > 1$: We have
      i. $|f_a^n(x)| \to \infty$ $\forall x \neq 0$
      ii. $f_a^{-n}(x) \to 0$ $\forall x \in \mathbb{R}$

      so 0 is a repelling fixed point with $W^s(0) = \{0\}$ and $W^u(0) = \mathbb{R}$.

   (c) $a = 1$: $f_1(x) = x$ so $\mathrm{Fix}(f_1) = \mathbb{R}$ with $\gamma_+(x) = x$ and $\omega(x) = \{x\}$ $\forall x \in \mathbb{R}$.

   (d) $a = -1 :$ $f_{-1}(x) = -x \Rightarrow f_{-1}^2(x) = x$ so $\mathrm{Per}_2(f_{-1}) = \mathbb{R}$ with $\gamma_+(x) = \{x, -x\}$ and $\omega(x) = \{x, -x\}$ $\forall x \neq 0$.

2. Find the ω-limit sets of the fixed and period 2 points of the logistic map $f_\mu(x) = \mu x(1 - x)$, $\mu > 0$ as found in example 3B.

   Recall from example 3B that

   $$\mathrm{Per}_1(f_\mu) = \begin{cases} \left\{0, \frac{\mu-1}{\mu}\right\} & \mu \neq 1 \\ \{0\} & \mu = 1 \end{cases} \qquad \mathrm{Per}_2(f_\mu) = \left\{0, \frac{\mu - 1}{\mu}, q_\mu^+, q_\mu^-\right\}$$

where $q_\mu^\pm = \dfrac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$. We therefore have

$$\omega(0) = \{0\}\,, \qquad \omega\left(\frac{\mu-1}{\mu}\right) = \left\{\frac{\mu-1}{\mu}\right\} \qquad \forall \mu > 0$$

and

$$\omega\left(q_\mu^+\right) = \omega\left(q_\mu^-\right) = \left\{q_\mu^+, q_\mu^-\right\} \qquad \mu > 3.$$

Note that $\omega(1) = \{0\}$

3. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$. Recall that

$$\mathrm{Per}(f) = \mathrm{Fix}(f) = \{0, 1, -1\}\,.$$

Find the stable and unstable sets of these fixed points.

Solution:

Using analysis we have

$$f^n(x) = x^{3^n} \to \begin{cases} 0 & \text{if} \quad |x| < 1 \\ \infty & \text{if} \quad x > 1 \\ -\infty & \text{if} \quad x < -1. \end{cases} \qquad f^{-n}(x) = x^{\frac{1}{3^n}} \to \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0. \end{cases}$$

(note: $x^{1/n} \to 1$ as $n \to \infty$ if $x > 0$.)

$$\begin{aligned} W^s(0) &= \{x : -1 < x < 1\}\,; & W^u(0) &= \{0\} \\ W^s(1) &= \{1\}\,; & W^u(1) &= \{x : x > 0\} \\ W^s(-1) &= \{-1\}\,; & W^u(-1) &= \{x : x < 0\} \end{aligned}$$

Note: We also have $\omega(x) = \{0\}$ $\forall x \in (-1, 1)$, and so $S = \{0\}$ is the only attractor for f.