Chapter 7: Limits and Continuity

7.1 Overview of Limits for Function of One Variable

- The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is critical difference.
- For a function of a single variable to have a limit, we need only check the values for the left-hand and right-hand limits.
- We recall that, if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x),$$

then $\lim_{x \to a} f(x)$ exists, and vice versa.

- On the other hand, if

$$\lim_{x \to a^-} f(x) = L_1 \neq \lim_{x \to a^+} f(x) = L_2,$$

then $\lim_{x \to a} f(x)$ does not exists, and vice versa.

For functions of two variables the situation is different.

7.2 Limits and Continuity for Functions of Two Variables

If $(x,y) \to L$ as (x,y) approaches (a,b) along every possible path that approaches (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

If $f(x,y) \to L_1$ as (x,y) approaches (a,b) along path P_1 but $f(x,y) \to L_2$ as (x,y) approaches (a,b) along path P_2, so

$$\lim_{(x,y)\to(a,b)} f(x,y) \textbf{ does not exist}.$$

Properties of Limits:

Linearity:

- $\displaystyle \lim_{(x,y)\to(x_0,y_0)} cf(x,y) = c \lim_{(x,y)\to(x_0,y_0)} f(x,y)$
- $\displaystyle \lim_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y)$

Products of functions:

- $\displaystyle \lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot \lim_{(x,y)\to(x_0,y_0)} g(x,y)$

Quotients of functions:

- $\displaystyle \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)}{\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y)}$

Example 1: Evaluate

(a) $\lim_{(x,y)\to(-1,0)} \left(xy^2 + x^3y + 5\right)$ Ans: 5

(b) $\lim_{(x,y)\to(3,4)} \dfrac{x-y}{\sqrt{x^2+y^2}}$ Ans: -1/5

Example 2: Evaluate

(a) $\lim_{(x,y)\to(1,2)} \dfrac{\left(x^2-1\right)\left(y^2-4\right)}{(x-1)(y-2)}$ Ans: 8

(b) $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x+y)}{x+y}$ Ans: 1

Example 3:

Determine whether or not

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^3+y^3}$$

exists, by examining the paths along the x-axis, y-axis, and also $y = x^2$.

Definition (Continuity)

A function *f* is continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

If *f* continuous at every point (a,b) in a region $\Re$, then *f* is continuous on $\Re$.

Example 4: Show that

(a) $f(x,y,z) = \ln(2x + y - z)$ is continuous at $(2,0,-1)$.

(b) $$g(x,y) = \begin{cases} \dfrac{x^2}{x^2 + y^2}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is discontinuous at $(0,0)$.

Example 5: Show that

$$g(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous at (0,0).

Example 6:

Extend the function $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$ to make it continuous at the origin.