Chapter 3 Multiple Integral

3.1 Double Integrals

3.2 Iterated Integrals

3.3 Double Integrals in Polar Coordinates

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- Triple Integrals in Cartesian Coordinates

- Triple Integrals in Cylindrical Coordinates

- Triple Integrals in Spherical Coordinates

3.5 Moments and Centre of Mass

3.1 Double Integrals

Recall: We think of the integral in two different ways.

In one way we interpret it as the area under the graph $y = f(x)$, while the *Fundamental Theorem of Calculus* enables us to compute this using the process of **anti-differentiation** – undoing the differentiation process.

We think of the area as

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x_k = \int_{a}^{b} f(x)dx$$

where the first sum is thought of as a limiting case, adding up the areas of a number of rectangles each of height $f(x_k)$, and width Δx_k.

This leads to the natural generalisation to several variables.
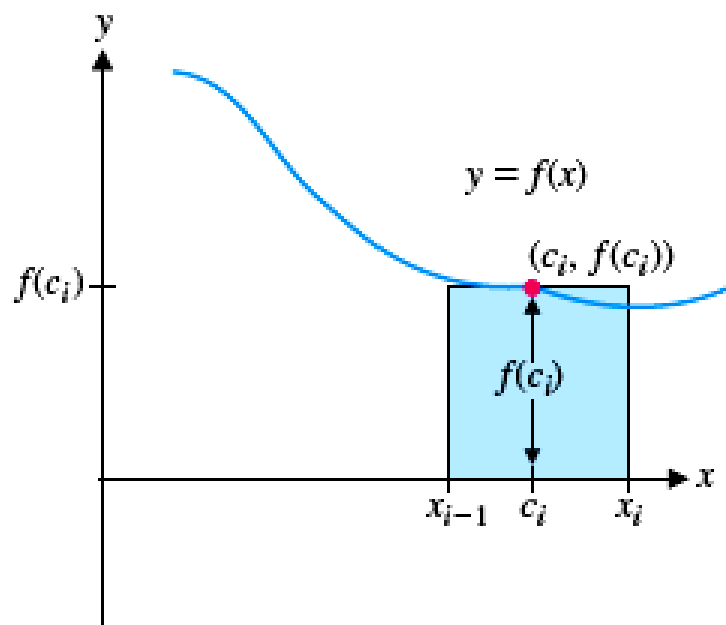
FIGURE 13.1a

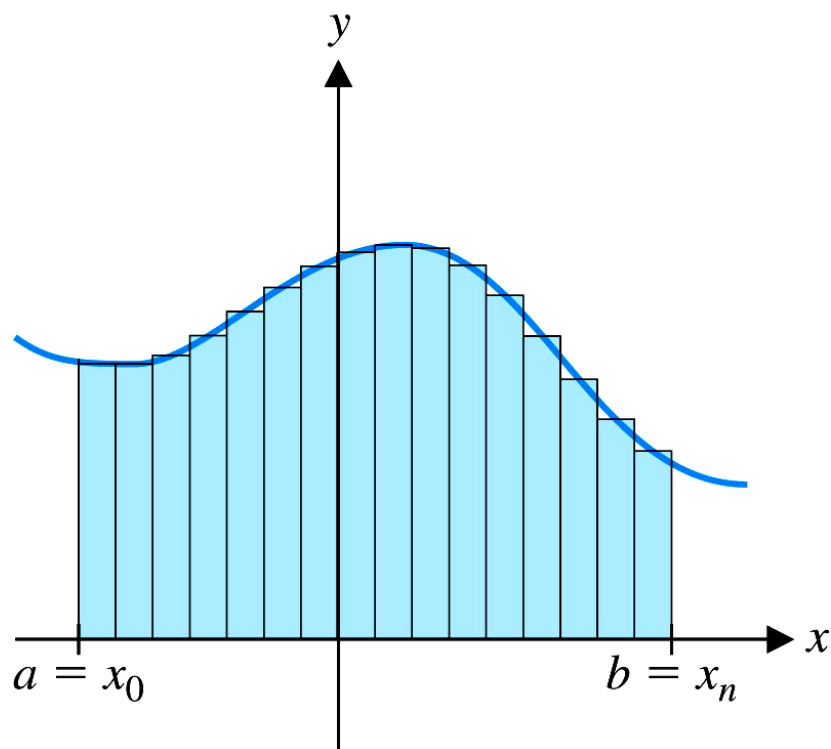Approximating the area on the
subinterval $[x_{i-1}, x_i]$


FIGURE 13.1b
Area under the curve

The double integral $\iint\limits_{R} f(x,y)dA$ has a similar interpretation in terms of volume except that the approximating elements will be **rectangular parallelepipeds** rather than rectangles.

A three-dimensional solid bounded by $z = f(x, y)$ and a region R in the xy-plane is approximated by a collection of boxes.

$z = f(x, y)$

(a)

(b)
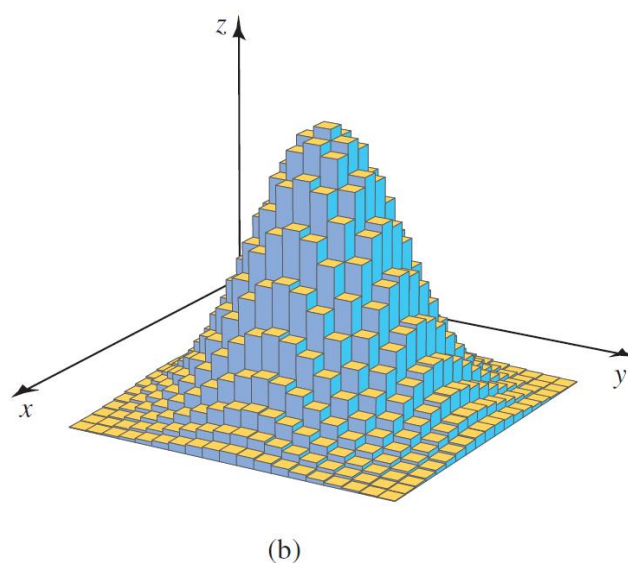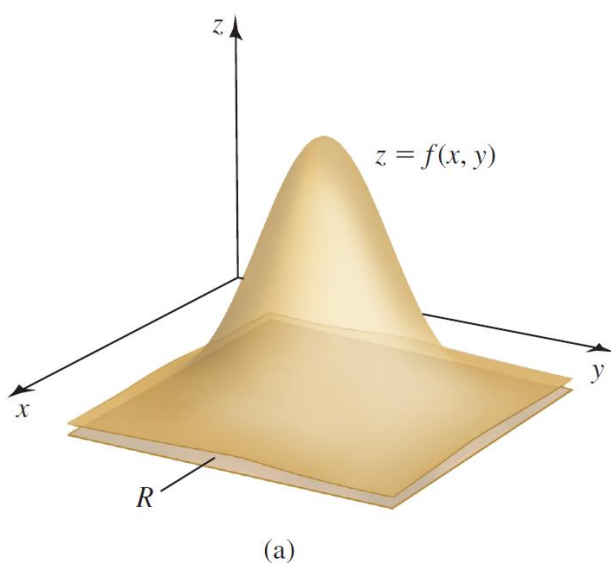
We first suppose that $f(x,y) \geq 0$. The graph of f is a surface with equation $z = f(x,y)$. Let σ be the solid that lies above R and under the graph of f.
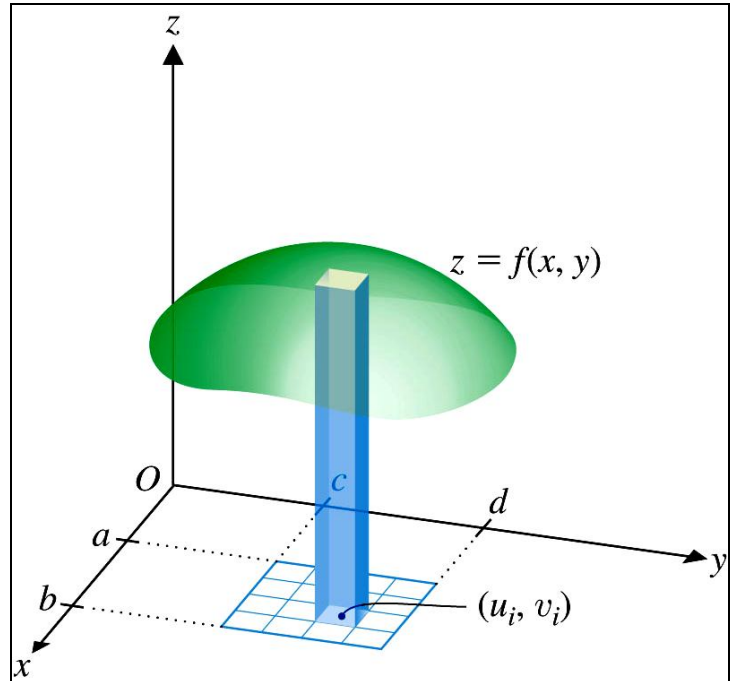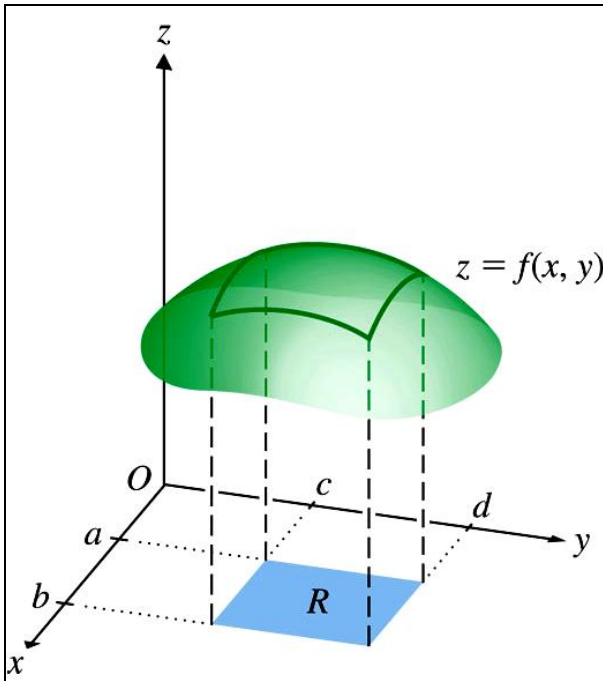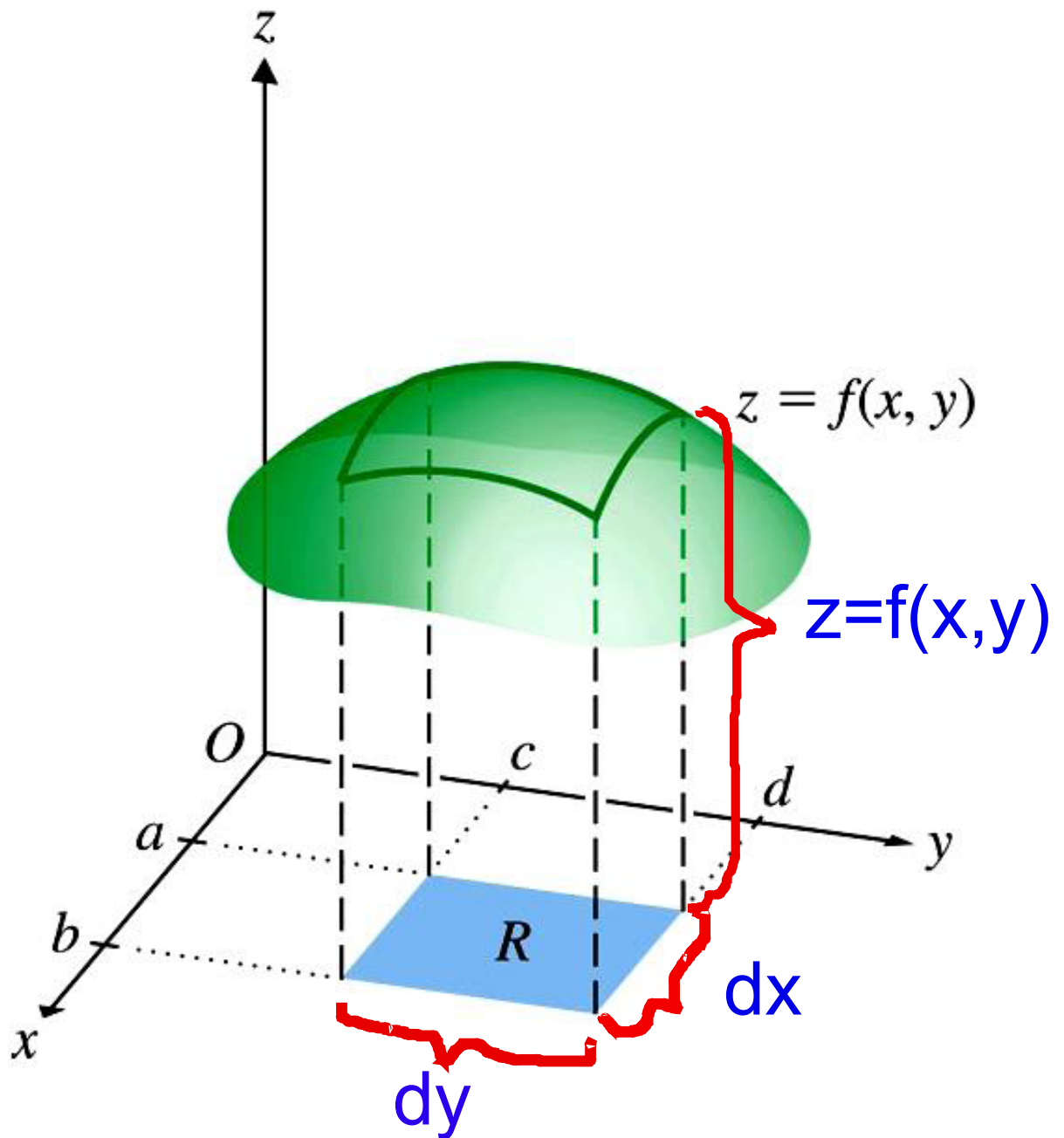
$z = f(x, y)$

$z = f(x, y)$

dx

dy

R

Area=dydx

Height = z

Volume of parallepiped = Area X Height
= z X dydx
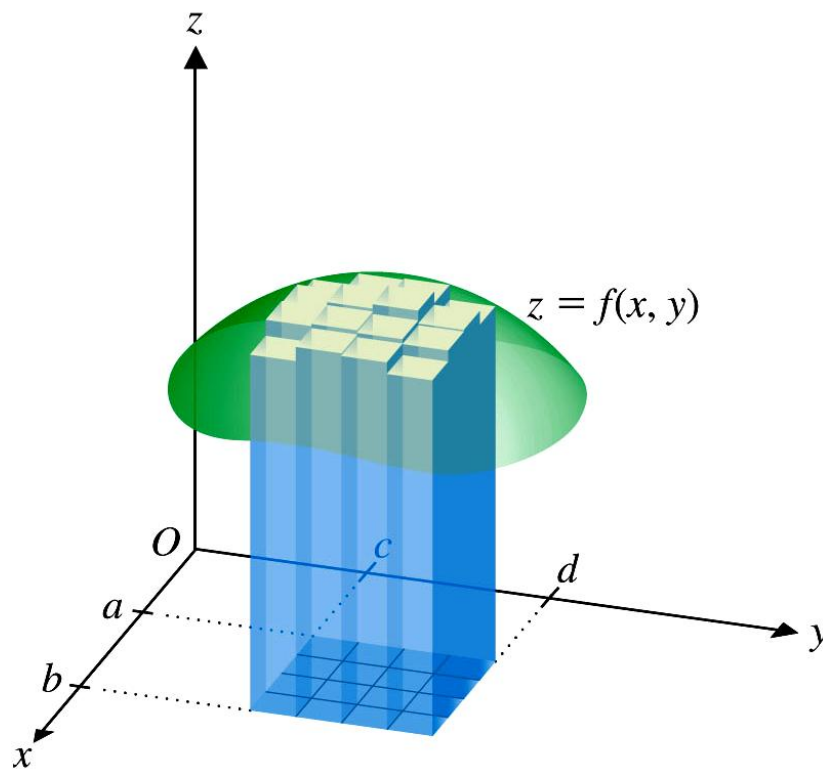
Our goal is to find the volume of σ.

♦ The first step of this process is to divide the region R into small rectangles.

♦ We can then compare the part of $z = f(x,y)$ that lies above the small rectangle, and this forms a thin box called rectangular parallelepiped.

♦ The volume of this parallelepiped is the area of the base times the height.

♦ Follow this procedure for all of the rectangles and add the volumes of the

corresponding parallelepipeds, we get an approximation of the total volume of σ:

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j) \Delta A$$

♦ Now, to find the area exactly, we simply make the boxes on the region *R* infinitely small (hence there are infinitely many). We do this by taking the limit of the above equation:

$$V = \lim_{i,j \to \infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j) \Delta A$$
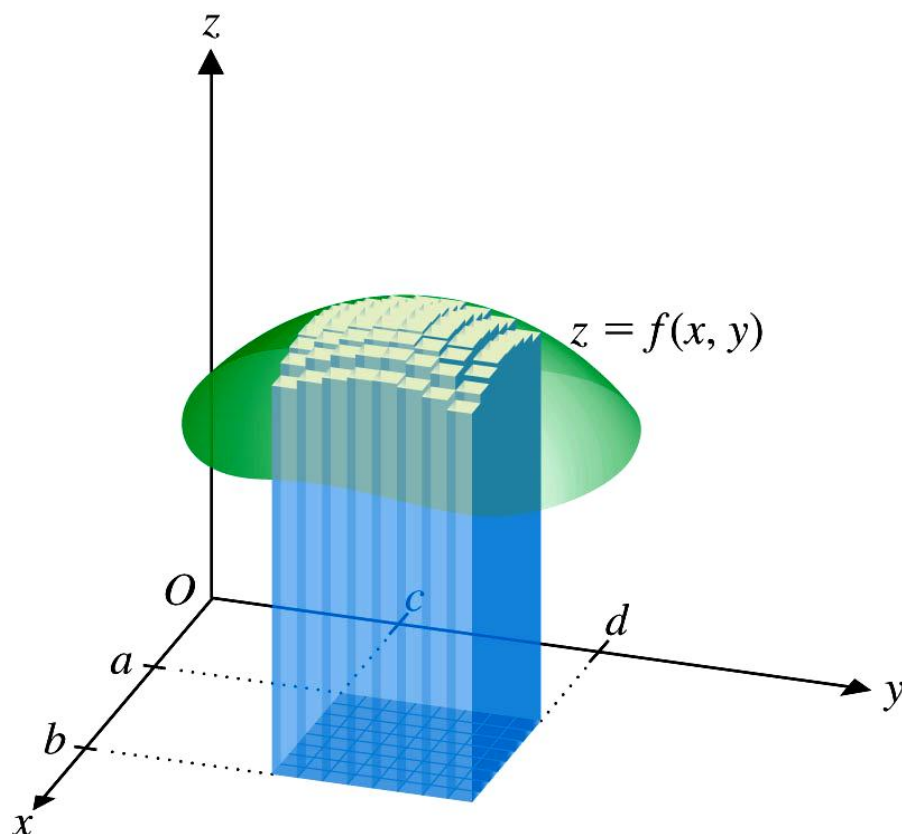
♦ By taking the limit above, we arrive at the definition of a double integral.
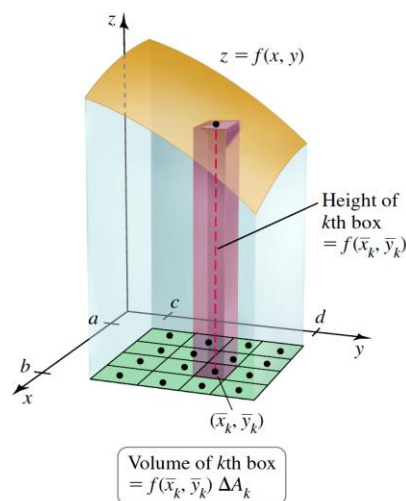
FIGURE 13.3
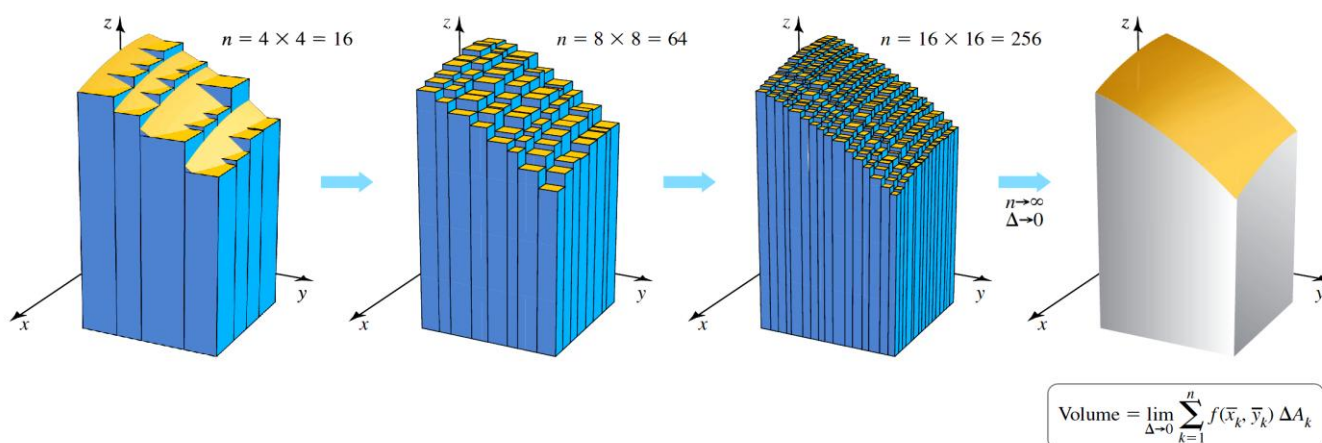
Definition 3.1

If f is a function of two variables that is defined on a region R in the xy-plane, then the **double integral of f over R** is given by

$$\iint\limits_{R} f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\Delta A$$

provided this limit exists, in which case f is said to be **integrable** over R.

Note

➢ The double integral of the surface $z = f(x,y)$ is the volume between the region R and below the surface.

➢ The sum:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j) \Delta A$$

is called the double **Riemann sum** and is used as an approximation to the value of the double integral.

The double integral inherits most of the properties of the single integral.

3.1.1 Properties of Double Integrals

1. constant multiple rule

$$\iint\limits_{R} cf(x,y)dA = c\iint\limits_{R} f(x,y)dA,\ c \text{ a constant}$$

2. linear rule

$$\iint\limits_{R} [f(x,y)+g(x,y)]dA$$
$$= \iint\limits_{R} f(x,y)dA + \iint\limits_{R} g(x,y)dA$$

3. subdivision rule

$$\iint\limits_{R} f(x,y)dA = \iint\limits_{R_1} f(x,y)dA + \iint\limits_{R_1} f(x,y)dA$$

4. dominance rule, if $f(x,y) \geq g(x,y)$

$$\iint\limits_{R} f(x,y)dA \geq \iint\limits_{R} g(x,y)dA$$

3.2 Iterated Integrals

3.2.1 Evaluating Double Integrals

♦ It is impractical to obtain the value of double integral from the definition. We evaluate the integrals by calculating two successive *single* integrals.

We use the notation $\int_{c}^{d} f(x,y)dy$ to mean that

x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This is called **partial integration** with respect to y.

$$A(x) = \int_{c}^{d} f(x,y)dy$$

Now we integrate the function A with respect to x from $x = a$ to $x = b$, we get:

$$\int_{a}^{b} A(x)dx = \int_{a}^{b} \left[\int_{c}^{d} f(x,y)dy \right] dx$$

This successive integration process is called *iterated integration*.

$$\iint f(x,y)\,dx\,dy = \int \left[\int f(x,y)\,dx \right] dy$$

$$\iint f(x,y)\,dy\,dx = \int \left[\int f(x,y)\,dy \right] dx$$

♦ These iterated integrals mean that we first integrate with respect to one variable (while holding the other fixed) and then integrating with respect to the other variable while holding the first one fixed.

♦ It is traditional to omit the brackets and write the iterated integral simply as

$$\iint f(x,y)\,dx\,dy$$

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Theorem 3.1 Fubini's Theorem

If $f(x,y)$ is continuous over the **rectangle**
$R: a \leq x \leq b, c \leq y \leq c$, then

$$\iint\limits_R f(x,y)dA = \int\limits_c^d \int\limits_a^b f(x,y)dxdy$$

$$= \int\limits_a^b \int\limits_c^d f(x,y)dydx$$

Example 3.1

Evaluate the integrals.

(a) $\int\limits_0^3 \int\limits_1^2 (1 + 8xy)\, dy\, dx$

(b) $\int\limits_1^2 \int\limits_0^3 (1 + 8xy)\, dx\, dy$

Compare (a) and (b). What can you say about the integration?

Prompts/Questions

- How are double integrals evaluated as iterated integrals?
  - Which theorem do you use?
- What is the inner integral?
  - Which variable is kept fixed?
- What is the outer integral?
- What integral rules & techniques do you

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Solution

(a) $\displaystyle\int_0^3\int_1^2(1+8xy)\,dy\,dx = \int_0^3\left[\int_1^2(1+8xy)\,dy\right]dx$

$\displaystyle = \int_0^3\left[y+4xy^2\right]_1^2 dx$

$\displaystyle = \int_0^3 1+12x\,dx$

$\displaystyle = x+6x^2\Big|_0^3 = 57$

(b) $\displaystyle\int_1^2\int_0^3(1+8xy)\,dx\,dy = \int_1^2\left[\int_0^3(1+8xy)\,dx\right]dy$

$\displaystyle = \int_1^2\left[x+4x^2y\right]_0^3 dy$

$\displaystyle = \int_1^2 3+36y\,dy$

$\displaystyle = 3y+18y^2\Big|_1^2 = 57$

Example 3.2

Compute

$$\iint\limits_{R} (2 - y)\, dA \text{ where } R$$
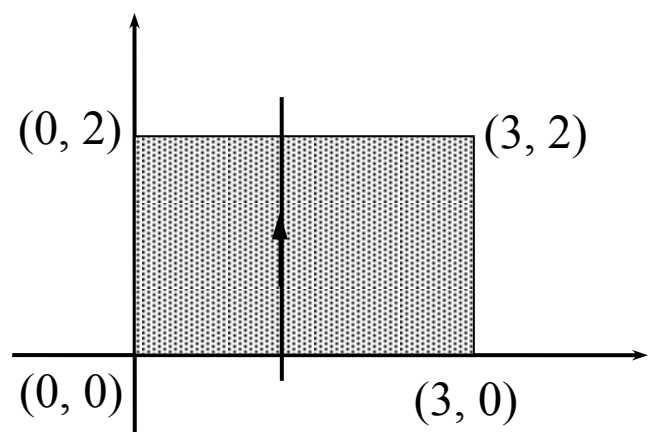
is a rectangle with vertices (0, 0), (3, 0), (3, 2) and (0, 2).

Prompts/Questions

- How do you write the integral as an iterated integral?
  - Can you sketch the region of integration?
  - What are the limits of integration
- Does the order of integration matter?

Solution

- ◆ Sketch the region of integration, *R*:
- ◆ Choose order of integration: fixed *x* (*y*-integration first, vertical arrow)

For each fixed *x* on the interval [0, 3], *y* ranges from 0 up to 2 and we get the double iterated integral,

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$$\iint_R 2 - y \, dA = \int_0^3 \int_0^2 (2-y) \, dy \, dx = \int_0^3 \left[2y - \frac{y^2}{2} \right]_0^2 dx$$

$$= \int_0^3 2 \, dx = 6$$

Example 3.3

Evaluate
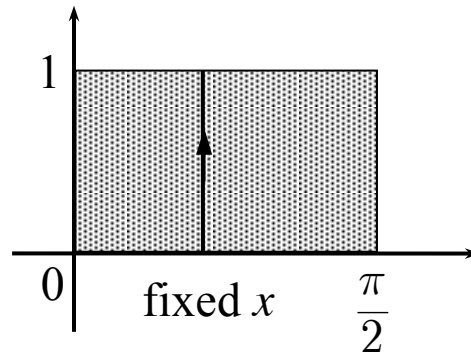$$\iint_R x \cos xy \, dA \text{ over the}$$
region R
$$0 \le x \le \frac{\pi}{2}, 0 \le y \le 1.$$

Prompts/Questions

- Can you sketch the region R ?
- How do you write the integral as an iterated integral?
  - ○ Which order of integration is easier?
  - ○ What are the limits of integration?
- What integration formulas do you use?

Solution

◆ Sketch the region R:

◆ Choose order of integration and set up the limits of integration: y-integration first, fixed x

$$\iint_R x\cos xy\,dA = \int_0^{\pi/2}\int_0^1 (x\cos xy)\,dy\,dx$$

$$= \int_0^{\pi/2}\left[x\frac{\sin xy}{x}\right]_0^1 dx$$

$$= \int_0^{\pi/2} \sin x\;dx = \left.-\cos x\right|_0^{\pi/2}$$

$$= 1$$

3.2.2 Nonrectangular Regions

We limit our study of double integrals to two basic types of regions: Type I and Type II.

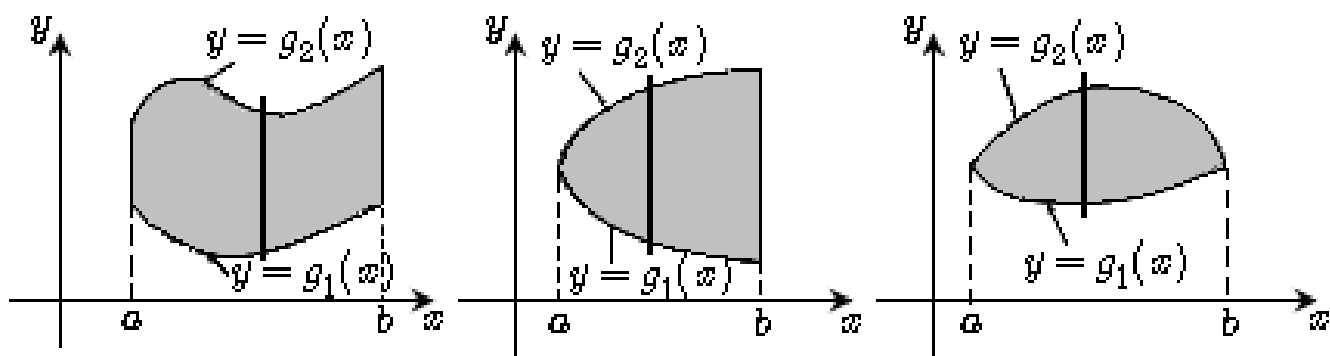# Definition 3.2

(a) A plane region R is said to be of **Type I** if it lies between the graphs of two continuous functions of x.

$$R = \left\{(x,y): a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\right\}$$

(b) A plane region R is said to be of **Type II** if it lies between the graphs of two continuous functions of y.
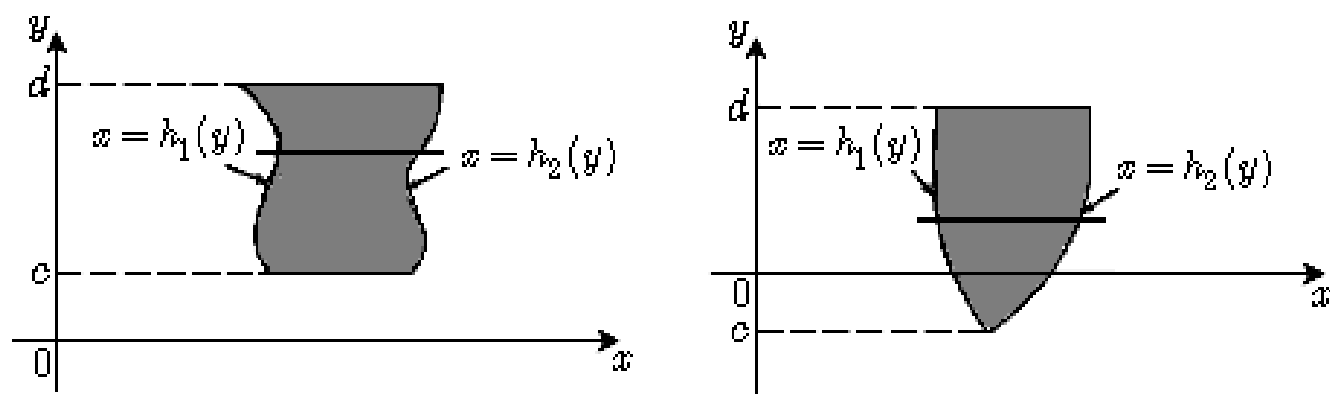
$$R = \left\{(x,y): h_1(y) \leq x \leq h_2(y), c \leq y \leq d\right\}$$

Type I Region - integrating first with respect to *y*

Type I (Vertical Strip): *x* fixed between *a* and *b*, *y* varies from $g_1(x)$ to $g_2(x)$.

Type II Region - integrating first with respect to *x*

Type II (Horizontal Strip): *y* fixed between *c* and *d*, *x* varies from $h_1(y)$ to $h_2(y)$.

Theorem 3.2

(a) If R is a **Type I** region, then

$$\iint\limits_{R} f(x,y)dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

(b) If R is a **Type II** region, then

$$\iint\limits_{R} f(x,y)dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

*Example 3.4a

Evaluate $\iint\limits_{R} (x + y) \, dA$ over the region R enclosed by the lines $y = 0$, $y = 2x$ and $x = 1$.

Solution

♦ Sketch the region: set up the limits of integration
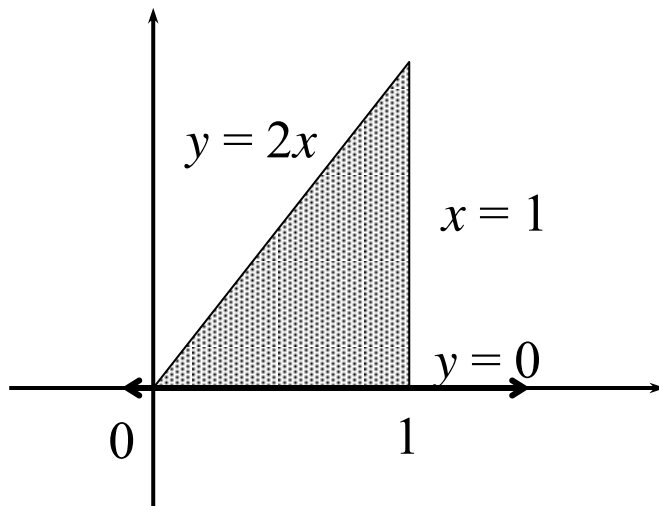
Choose order of integration: Type I, fixed x

$$\iint_R (x + y)\, dA = \int_0^1 \int_0^{2x} (x + y)\, dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=2x} dx = \int_0^1 4x^2\, dx$$

$$= \frac{4}{3} x^3 \Big|_{x=0}^{x=1} = \frac{4}{3}$$

Alternatively, reversing the order of integration: Type II, fixed y
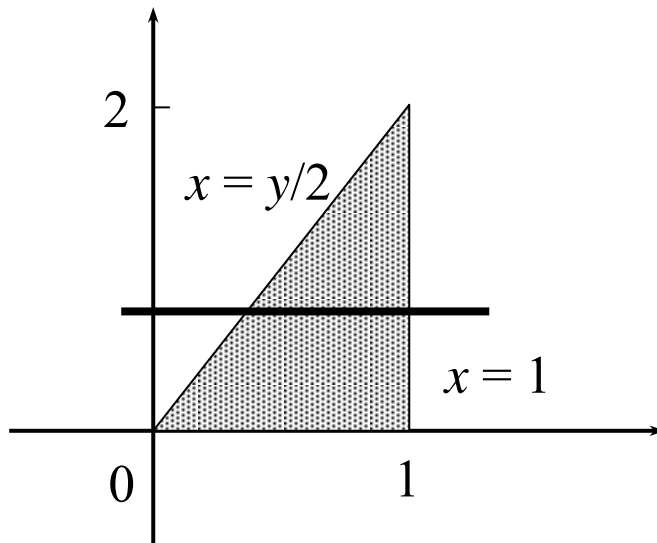
$$\iint\limits_R (x + y)\, dA = \int\limits_0^2 \int\limits_{y/2}^1 (x + y)\, dxdy$$

$$= \int\limits_0^1 \left[\frac{x^2}{2} + xy\right]_{x=y/2}^{x=1} dy$$

$$= \int\limits_0^1 \left[\frac{1}{2} + y - \frac{5y^2}{8}\right] dy$$

$$= \frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24}\Bigg|_{y=0}^{y=2} = \frac{4}{3}$$

Example 3.4b

Evaluate

$$\iint\limits_{R} (2x + y^2)\, dA \text{ over}$$

the region R in the first quadrant bounded by the axes and $y^2 = 1 - x$.

Prompts/Questions

- How do write the integral as an iterated integral?
  - o Can you identify and sketch the region of integration?
  - o What are the limits of integration?
- Does the order of integration matter?
- What integral rules & techniques do you know?

Solution

We know the region R, what we want is to determine the limits of the iterated integrals.

♦ Sketch the region R.

♦ Choose order of integration:

Type I, fixed x (vertical strip, y-integration first)
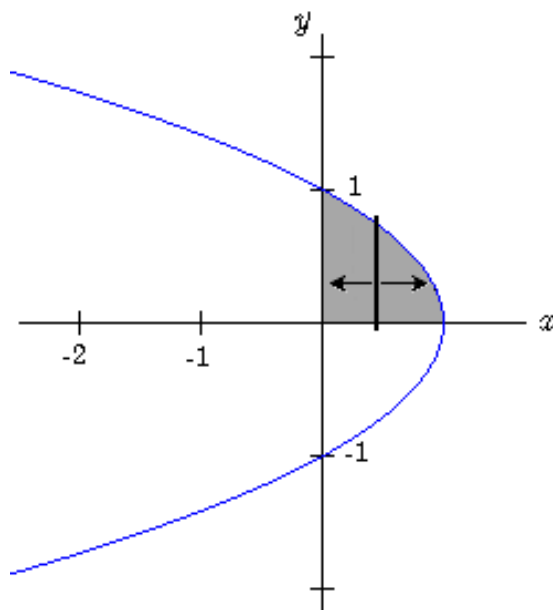
♦ Determine the limits of integration:

Recognise that for each fixed x on the interval $[0, 1]$, y ranges from 0 up to $\sqrt{1-x}$ and we get the iterated double integral,

$$\iint\limits_R (2x + y^2)\, dA = \int_0^1 \int_0^{\sqrt{1-x}} (2x + y^2)\, dy\, dx$$

$$= \int_0^1 \left[2xy + \frac{y^3}{3} \right]_0^{\sqrt{1-x}} dx = \int_0^1 \left(2x\sqrt{1-x} + \frac{(\sqrt{1-x})^3}{3} \right) dx$$

$$= \cdots = \frac{2}{3}$$

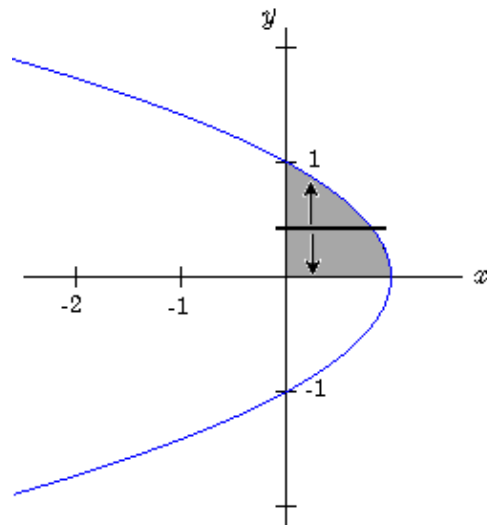Alternatively, choosing the other order of integration: Type II, fixed *y* (horizontal strip, *x*-integration first)

For each fixed *y* on the interval [0, 1], *x* ranges from 0 over to $1 - y^2$ and we get the iterated integral,

$$\iint\limits_{R} (2x + y^2)\,dA = \int\limits_{0}^{1} \int\limits_{0}^{1-y^2} (2x + y^2)\,dx\,dy$$

$$= \int_{0}^{1} \left[x^2 + y^2 x \right]_{0}^{1-y^2} dy \; = \int_{0}^{1} 1 - y^2 \, dy$$

$$\vdots$$

$$= \frac{2}{3}$$

For Type I, the computation involves some difficult integrals. Thus, in this case integrating in the order *dxdy* (Type II) is more convenient.

Note

➢ Unless the limits are constants, you cannot simply swap the integral and limit signs around. You have to draw out the entire region and see how it changes.

Example 3.5

Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_x^1 y^2 e^{xy} \, dy\,dx$$

Solution

Identify the region of integration *R*:

♦ Read the limits of integration: *y*-integration first

Integral is of Type I - for each fixed *x* on the interval [0, 1], *y* runs from $y = x$ up to $y = 1$.

♦ Sketch the region *R*: *Complete the solution…*

♦ Reverse order of integration: Type II, fixed *y* (horizontal strip, *x*-integration first)

For each fixed *y* on the interval [0, 1], *x* runs from $x = 0$ over to $x = y$. We obtain,

$$\int\limits_{0}^{1}\int\limits_{x}^{1} y^2 e^{xy}\, dy dx = \int\limits_{0}^{1}\int\limits_{0}^{y} y^2 e^{xy}\, dx dy$$

$$= \int_0^1 y^2 \frac{e^{xy}}{y}\bigg|_0^y \, dy = \int_0^1 ye^{y^2} \, dy$$

$$= \frac{1}{2}e^{y^2}\bigg|_0^1 = \frac{1}{2}(e-1)$$

There are integrals that can be evaluated **only** in one order and is impossible to do the integral in the other order. See example below.

Example 3.6

Evaluate $\displaystyle\int_0^4 \int_{x/2}^2 e^{y^2} \, dy\,dx$.

Prompts/Questions

- The integral is impossible to evaluate in the given order. Why?
- How do you reverse the order of integration?
  - Can you identify and sketch the region of integration?
  - What are the limits of integration

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Solution

In the given order, the integral cannot be evaluated because the integrand e^{y^2} has no antiderivative.

◆ Sketch the region R: *Complete the solution…*

Given order: y-integration first	Reversed order: x-integration first

$$\int_0^4 \int_{x/2}^2 e^{y^2} \, dy \, dx = \int_0^2 \int_0^{2y} e^{y^2} \, dx \, dy = \int_0^2 x e^{y^2} \Big|_0^{2y} \, dy$$

$$\cdots = \int_0^2 2y e^{y^2} \, dy = e^4 - 1$$

Example 3.7

Construct TWO examples of double integrals that are readily evaluated by integrating in one order but not in the reverse order.

Prompts/Questions

- How do you know the integral is easily evaluated in one particular order?

3.2.3 Double Integral as Area and Volume

Definition 3.3

(a) The **area** of the region R in the xy-plane is given by

$$A = \iint\limits_R dA$$

(b) If f is continuous and $f(x,y) \geq 0$ on the region R, the **volume** of the solid under the surface $z = f(x,y)$ above the region R is given by

$$V = \iint\limits_R f(x,y)\, dA$$

*Example 3.6

Find the area of the region bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

Sketch the region:
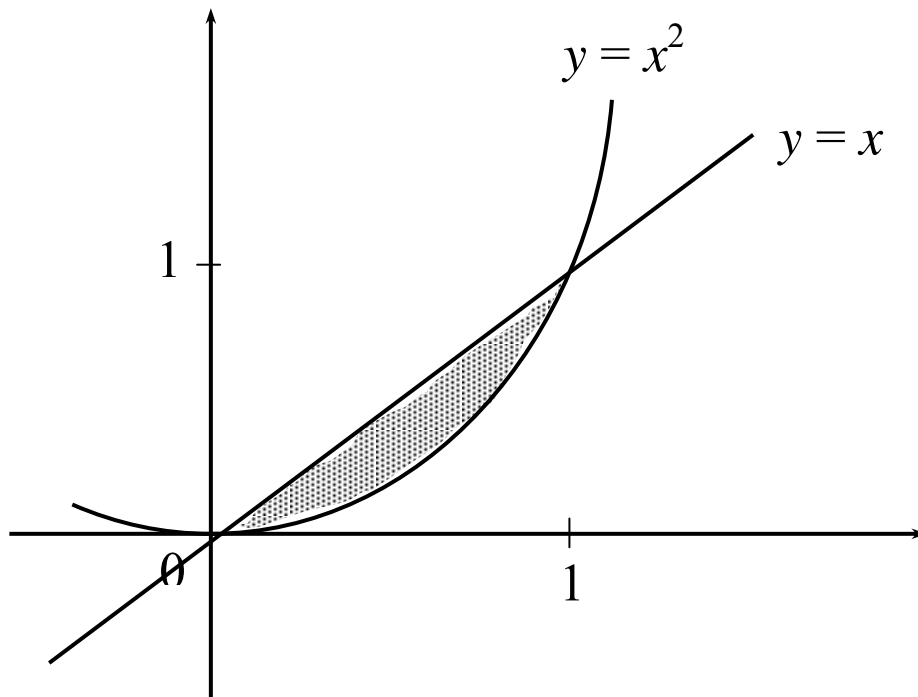
Order of integration: Type I, fixed x

$$\text{Area} = \int_0^1 \int_{x^2}^x \; dydx = \int_0^1 [y]_{x^2}^x \; dx$$

$$= \int_0^1 x - x^2 \; dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \, \text{unit}^2$$

Example 3.7

Find the area of the region enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Example 3.7a

Find the area of the region bounded by the graphs $y = x^2$ and the line $y = x + 2$.

Example 3.8

Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Example 3.8a

Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$ and $z = 0$.

Example 3.8b

Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2 - y^2$, $y = 2 - 2x^2$, $x = 0$, $y = 0$ and $z = 0$.

Example 3.9

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$ and $z = 0$.
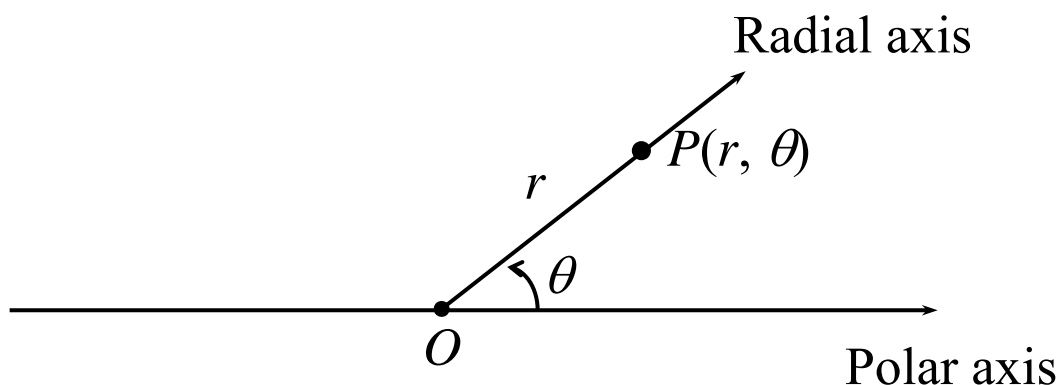
Solution

Volume, $V = \iint\limits_R z \, dA = \int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - y \, dy \, dx$

$= \int\limits_{-2}^{2} 4y - \dfrac{y^2}{2} \Bigg]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$

$= \int\limits_{-2}^{2} 8\sqrt{4-x^2} \, dx$

$= 32\int \cos^2 \theta \, d\theta$

$= 16\left[\dfrac{\sin 2\theta}{2} + \theta \right]$

$= 16\left[\left(\dfrac{x}{2}\right)\dfrac{\sqrt{4-x^2}}{2} + \sin^{-1}\left(\dfrac{x}{2}\right) \right]_{-2}^{2}$

$\therefore V \quad = 16\pi \, \text{unit}^3$

3.3 Double Integral in Polar Form

3.3.1 Polar Coordinates System

A polar coordinate system consist of a fixed point O called the origin or **pole** and a line segment starting from the pole called the **polar axis**.

r – radial coordinate
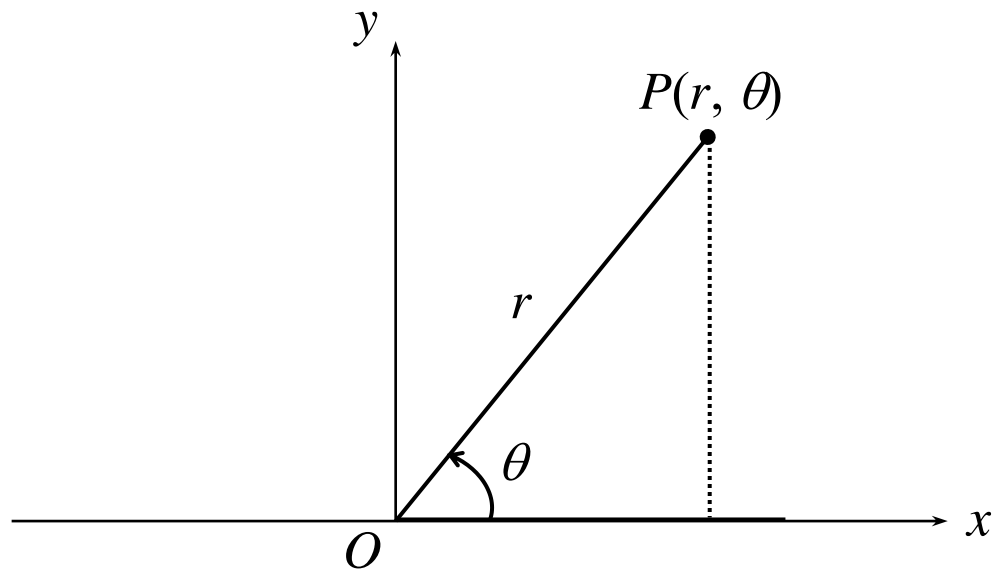
θ – polar angle

Definition 3.3

Polar coordinates of a point P is written as (r, θ) where r is the distance of P from the pole and θ is the angle measured from the polar axis to the line OP (radial axis).

3.3.2 Relationship between Polar and Cartesian Coordinates
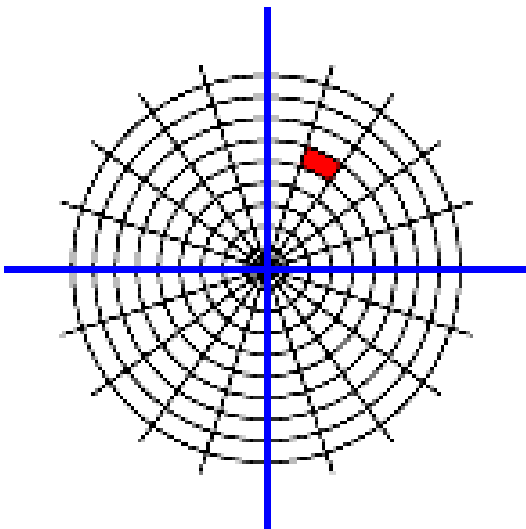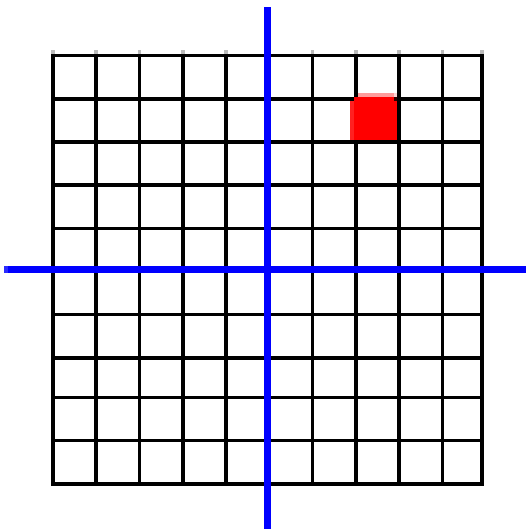

$$x = r \cos \theta \qquad y = r \sin \theta,$$

$$x^2 + y^2 = r^2 \qquad \tan \theta = \frac{y}{x}$$

Note

(i) Polar coordinate of a point is not unique.

(ii) θ is positive in an anticlockwise direction, and negative if it is taken clockwise.

(iii) A point $\left(- r, \theta\right)$ is in the opposite direction of point $\left(r, \theta\right)$.

Polar Grid
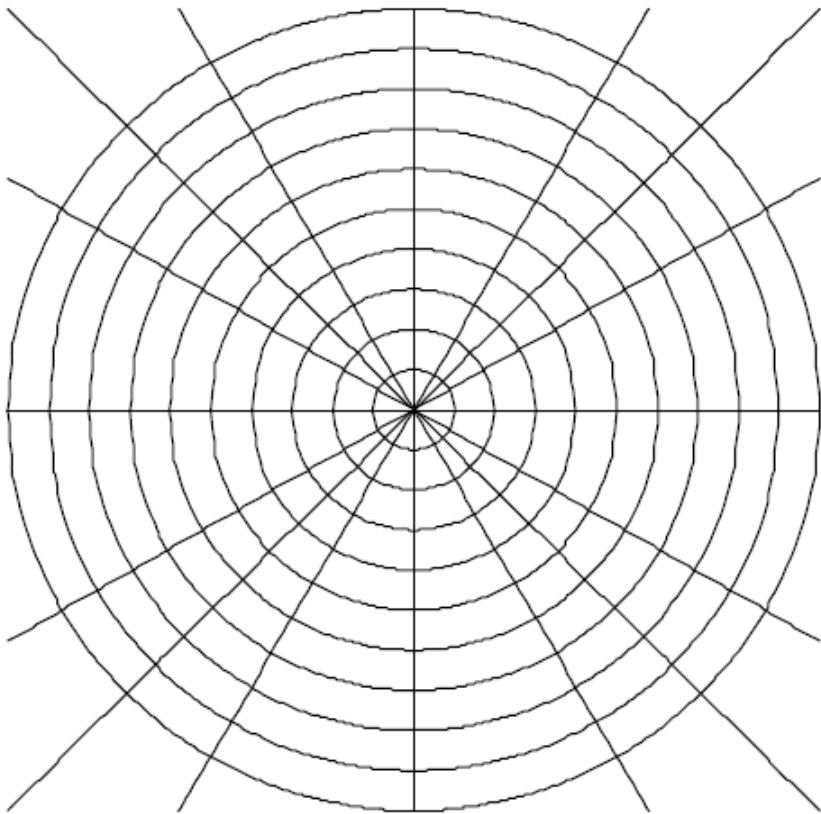
3.3.3 Integrals in Polar Coordinates

If R is a circular region (involves $x^2 + y^2$), it is easily described using polar coordinates.

♦ Divide the region into polar rectangles.
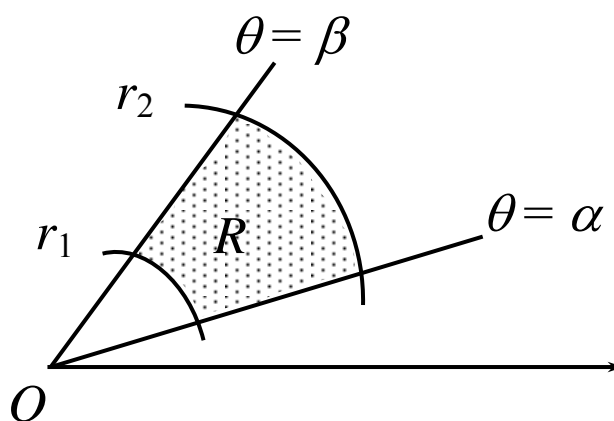
♦ Find the area of typical polar rectangle:
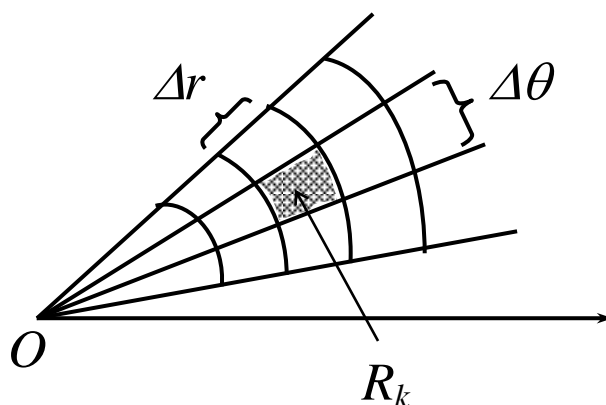
$\Delta A_k =$ area of large sector − area of small sector

$$= \frac{\Delta \theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = r_k \Delta r_k \Delta \theta_k$$

Alternatively:
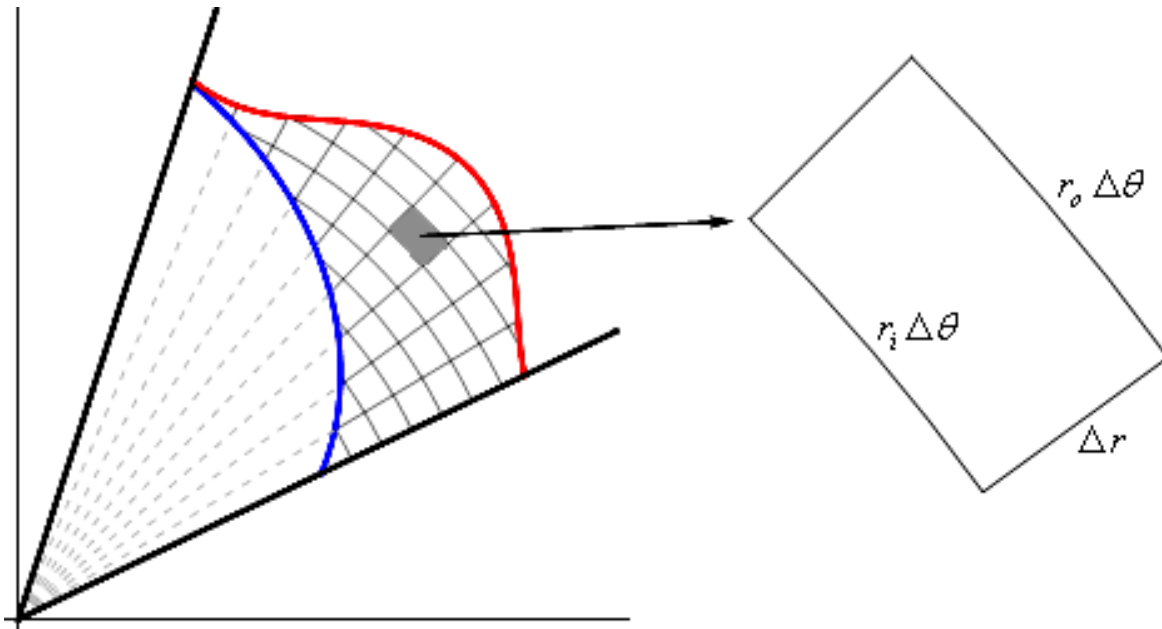
If the mesh is small enough, we can assume that,

$$r_0 \approx r_1 = r$$

and with this assumption we can also assume that our polar slab is close enough to a rectangle,

$$\Delta A \approx r \Delta\theta \Delta r$$

Thinking of volume, we make the equation $z = f(r\cos\theta, r\sin\theta)$, thus the Riemann sum can be written as:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*)\, r^* \Delta r \Delta \theta$$

Taking the limit we have the actual volume,

$$\iint_R f(x,y)\, dA = \iint_R f(r,\theta)\, r\, dr d\theta$$

A version of Fubini's Theorem now says that the integral can be evaluated by iteration with respect to r and θ.

Theorem 3.2

Let R be a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$. If $f(r,\theta)$ is continuous on R, then

$$\iint\limits_{R} f(x,y)\, dA = \int\limits_{\theta=\alpha}^{\theta=\beta} \int\limits_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

(a) (b)
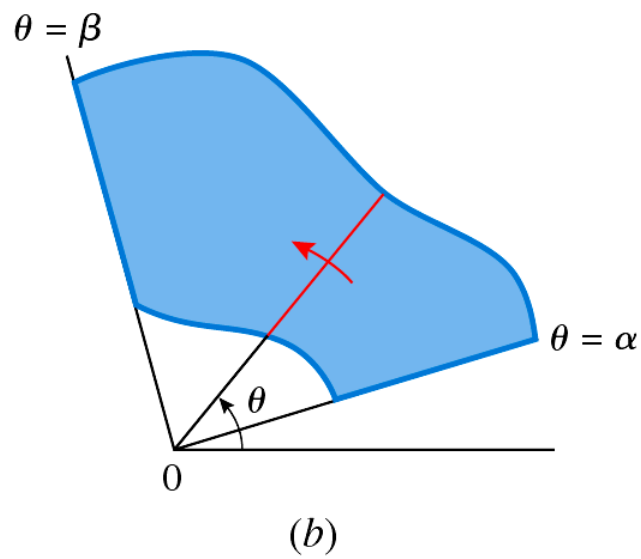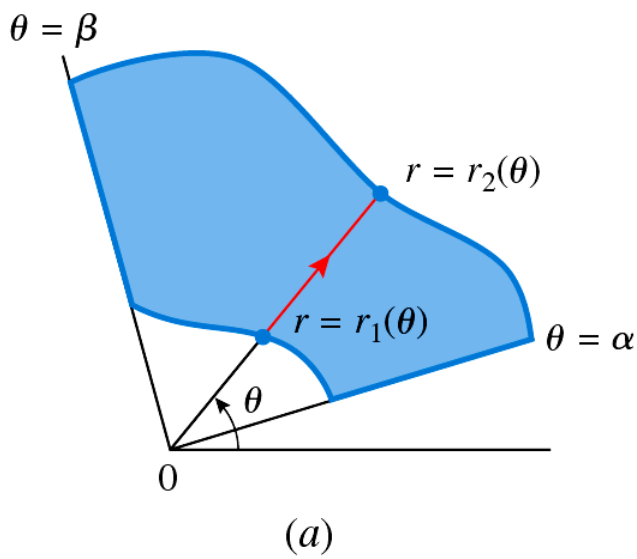
3.3.4 Finding limits of Integration

Example 3.10

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardiod $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

Step 1: Sketch R

$\theta = \pi/2$

$r = 1 + \cos \theta$

$r = 1$

1

2

$\theta = -\pi/2$
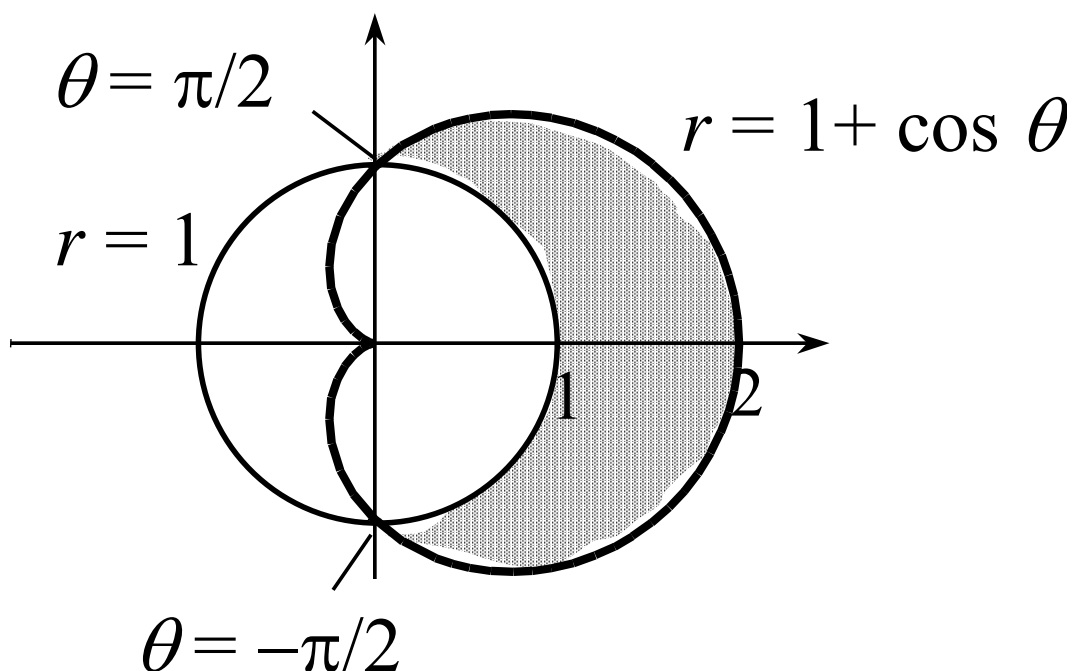
Step 2: the *r*-limits of integration

A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$.

Step 3: the *θ*-limits of integration

The rays from the origin that intersect R run from $\theta = -\dfrac{\pi}{2}$ to $\theta = \dfrac{\pi}{2}$.

The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r,\theta)\, r \, dr d\theta$$

Note

We may, of course, integrate first with respect to θ and then with respect to r if this is more convenient.

3.3.5 Changing Cartesian Integrals into Polar Integrals

The procedure for changing Cartesian integral $\iint\limits_{R} f(x,y)\,dA$ into a polar integral has two steps.

Step 1: Substitute $x = r\cos\theta$, $y = r\sin\theta$ and replace $dxdy$ by $r\,drd\theta$ in the Cartesian integral.

Step 2: Supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint\limits_{R} f(r,\theta)\,dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

Example 3.11

Evaluate $\displaystyle\iint\limits_{R} (x^2 + y^2 + 1)\, dA$ where R is the region inside the circle $x^2 + y^2 = 4$.

Solution

We evaluate the integral in polar form.

KNOW: $x^2 + y^2 = r^2$

Region R: $x^2 + y^2 = 4 \Rightarrow r^2 = 4$ or $r = 2$
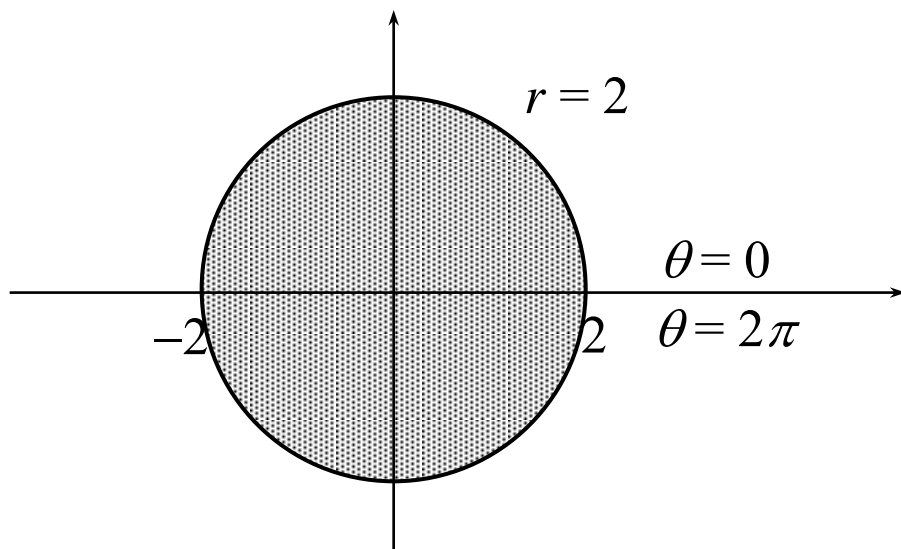
$$\iint\limits_{R} (x^2 + y^2 + 1)\, dA = \int\limits_{0}^{2\pi} \int\limits_{0}^{2} (r^2 + 1)\, r\, dr d\theta$$

Example 3.12

Evaluate $\iint\limits_R x \, dA$ where R is the region bounded above by the line $y = x$ and below by the circle $x^2 + y^2 - 2y = 0$.

Example 3.12a

To evaluate the integral

$$\iint_R r^2 \, dr \, d\theta$$

where R is the region in the xy-plane bounded by $r = 2\cos\theta$, we obtain

$$\int_0^\pi \int_0^{2\cos\theta} r^2 \, dr \, d\theta = \int_0^\pi \left.\frac{r^3}{3}\right|_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_0^\pi \cos^3\theta \, d\theta = \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^\pi = 0$$

Alternatively we can set up the integral as

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left.\frac{r^3}{3}\right|_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta = \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_{-\pi/2}^{\pi/2} = \frac{32}{9}$$

Both of these answers cannot be correct. Which procedure is correct and why?

Area between the lines $\theta = \alpha$, $\theta = \beta$ and the curves $r = f(\theta)$, $r = g(\theta)$.

$r = f(\Theta)$

$r = g(\Theta)$
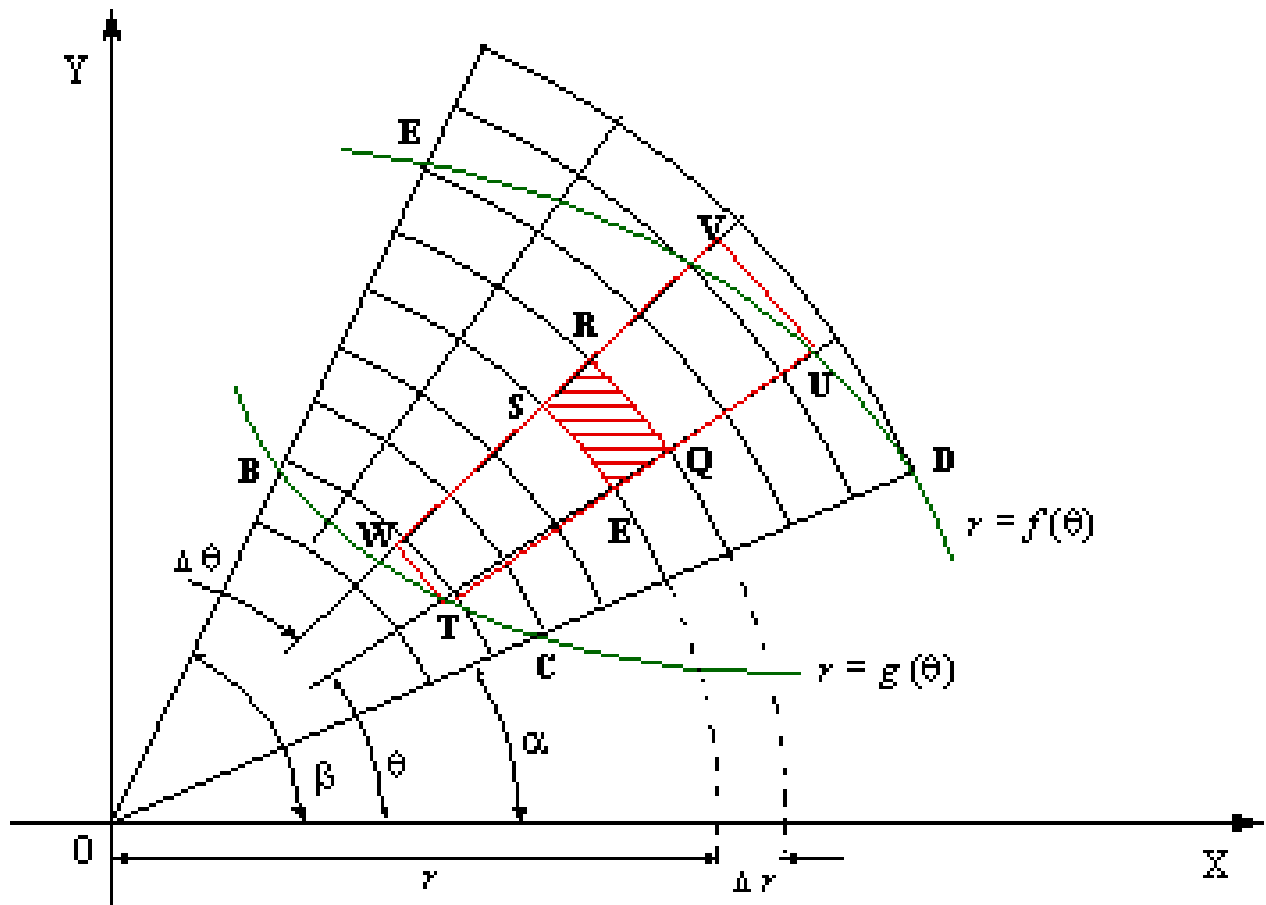
$$\iint\limits_{R} dA = \int\limits_{\alpha}^{\beta} \int\limits_{g(\theta)}^{f(\theta)} r\, dr\, d\theta$$

Area between the circles $r = a$, $r = b$ and the curves $\theta = \Psi(r)$, $\theta = \varphi(r)$.
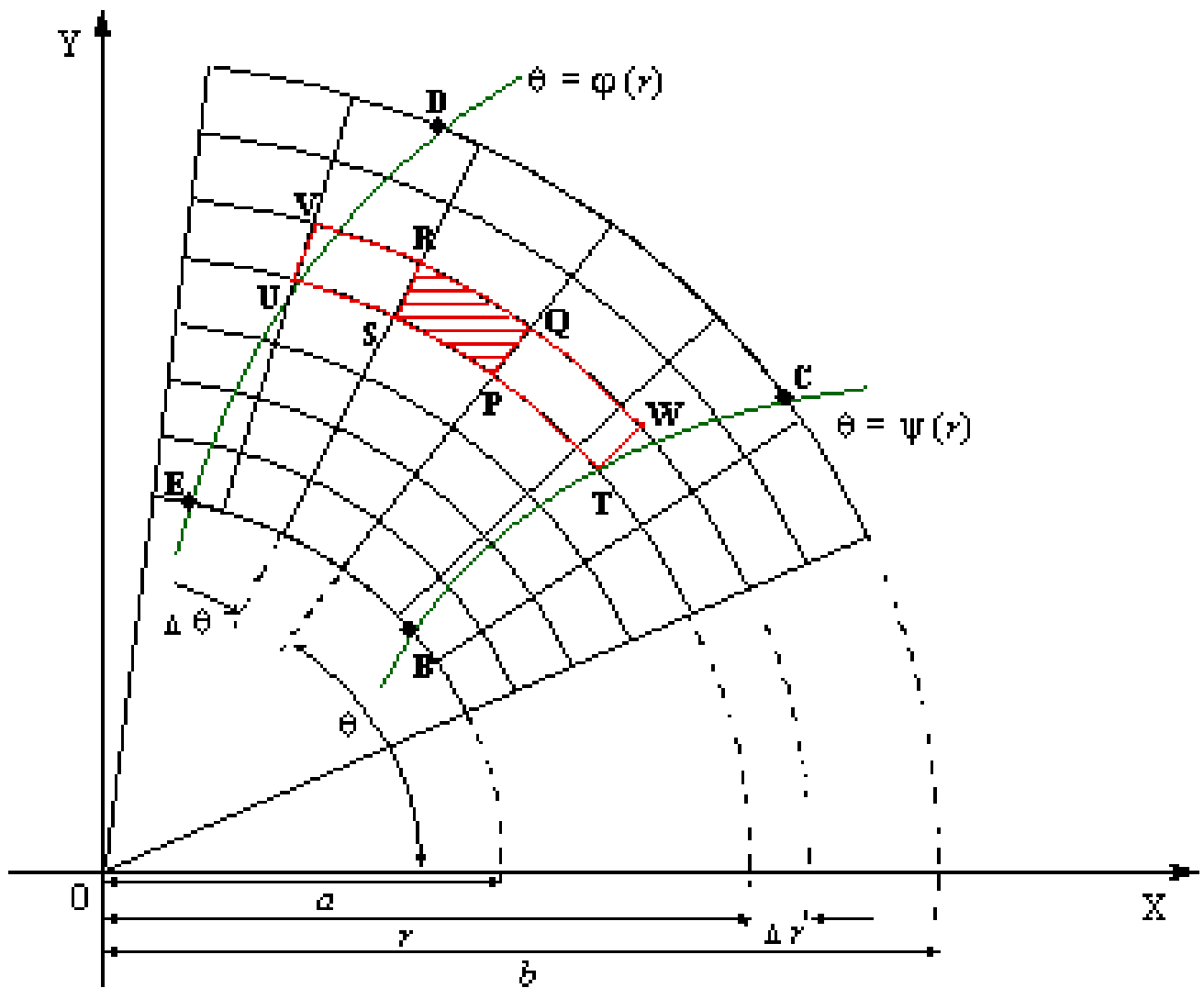
$$\iint\limits_{R} dA = \int\limits_{a}^{b} \int\limits_{\Psi(r)}^{\varphi(r)} r \, d\theta \, dr$$

Example 3.13

Use polar double integral to find the area enclosed by the three-petal rose $r = \sin 3\theta$.

Solution

Sketch the region:

$r = \sin 3\theta$
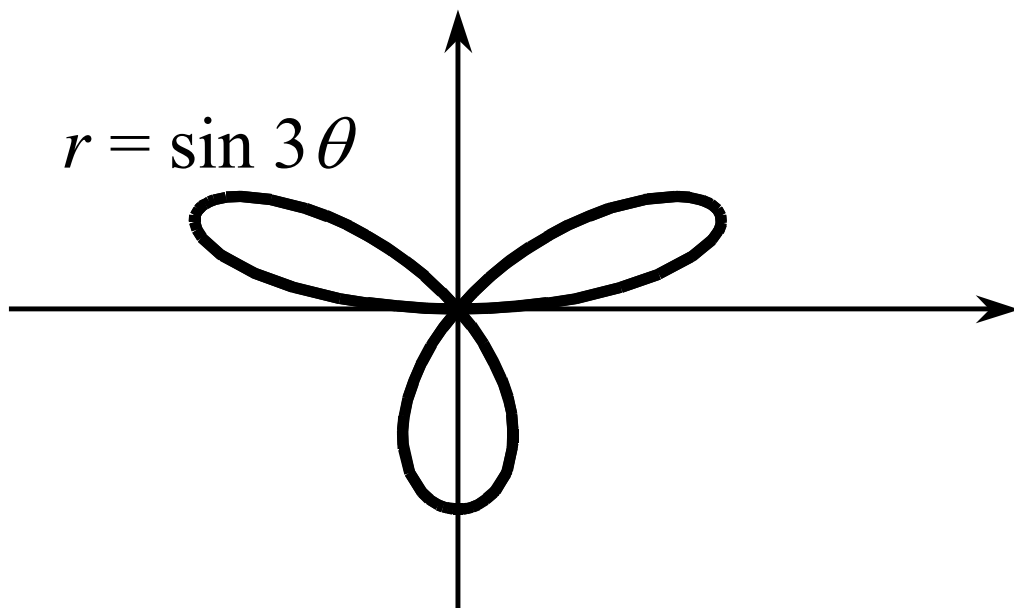
KNOW: Area, $A = \displaystyle\iint_R dA$

The graph is symmetry, so we will calculate the area of the petal in the first quadrant and multiply by 3.

$$A = 3\iint_{R_1} dA = 3\int_0^{\pi/3} \int_0^{\sin 3\theta} r\, dr d\theta$$

Example 3.14

Find the area bounded by the polar axis, part of the spiral $r\theta = 2$ and between the graphs $r = 2$ and $r = 3$.

Solution

♦ Sketch the region of integration. *Complete the solution...*

♦ Determine limits of integration: choose order of integration

Choose θ-integration first. For each fixed r between 2 and 3, θ varies from $\theta = 0$ over to $\theta = \dfrac{2}{r}$.

♦ Set up the integral and evaluate:

$$\therefore \text{ Area} = \iint_{R_1} dA = \int_2^3 \int_0^{2/r} r \, d\theta dr = \cdots = 2$$

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Alternatively, r-integration first will give us the integral,

$$\text{Area} = \int\limits_{0}^{2/3} \int\limits_{2}^{3} r\, dr d\theta + \int\limits_{2/3}^{1} \int\limits_{2}^{2/\theta} r\, dr d\theta = \cdots = 2$$

Example 3.15 (Example 3.9 revisited)

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$ and $z = 0$.

Example 3.15a

Find the volume of the solid under the surface $z = e^{x^2 + y^2}$ and above the region $1 \leq r \leq 3,\ 0 \leq \theta \leq \dfrac{\pi}{4}$.

Example 3.15b

Find the volume of the solid bounded by the graphs $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$.

3.4 Triple Integral

Focus of Attention

➢ Definition as a limit of Riemann sum – interpretation as volume of solid

➢ How are triple integrals evaluated as iterated integrals?

➢ Does the order of integration matter?

➢ How are the limits of integration determined?

➢ How is triple integral use to find volume?

➢ How do you transform a triple integral in rectangular coordinates into a triple integral in cylindrical coordinates or spherical coordinates?

➢ What do you look for when considering using cylindrical coordinates or spherical coordinates?

Recall that

♦ A single integral $\int f(x)\,dx$ is evaluated over a closed interval on the x-axis.

♦ A double integral $\iint\limits_{R} f(x,y)\,dA$ is evaluated over a closed bounded region in the plane.

and in essentially the same way

♦ A triple integral $\int\int\int\limits_{G} f(x,y,z)\,dV$ is evaluated over a closed, bounded solid region in $\mathbb{R}^3$.
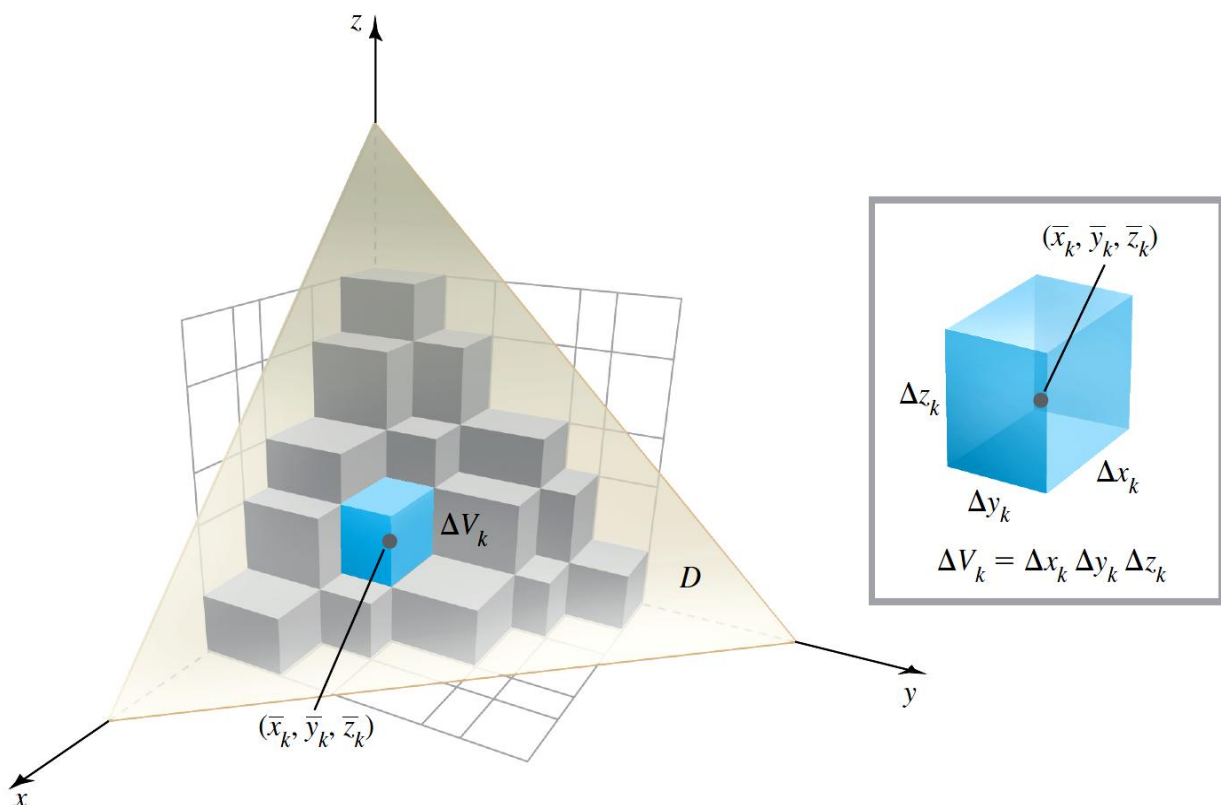
Definition 3.4

If f is a function defined over a closed, bounded solid region G, then the triple integral of f over G is defined as

$$\int\int\int_G f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

The properties of triple integrals are analogous to those of double integrals.

♦ Constant Multiple Rule

♦ Sum Rule

♦ Dominance Rule

♦ Subdivision/Additivity Rule

3.4.1 Iterated Integration

Just as for double integrals, the practical method for evaluating triple integrals is to expressed them as iterated integrals as in the following theorem:

Theorem 3.3

If $f(x, y, z)$ is continuous over a rectangle solid G: $a \leq x \leq b$, $c \leq y \leq d$, $k \leq z \leq l$, then the triple integral may be evaluated by the iterated integral

$$\iiint\limits_{G} f(x, y, z)\, dV = \int_{k}^{l} \int_{c}^{d} \int_{a}^{b} f(x, y, z)\, dx\,dy\,dz$$

The iterated integration can be performed in any order (with appropriate adjustments) to the limits of integration:

$$dx\, dy\, dz \qquad dx\, dz\, dy$$

$$dy\, dx\, dz \qquad dy\, dz\, dx$$

$$dz\, dy\, dx \qquad dz\, dx\, dy$$

Example 3.15

Evaluate $\iiint\limits_{G} z^2 y e^x \, dV$, over the rectangular box G defined by

$$0 \le x \le 1, \; 1 \le y \le 2, \; -1 \le z \le 1$$

Solution

We shall evaluate the integral in the order $dx \, dy \, dz$.

$$\iiint\limits_{G} z^2 y e^x \, dV = \int\limits_{-1}^{1} \int\limits_{1}^{2} \int\limits_{0}^{1} z^2 y e^x \, dx dy dz$$

$$= \int\limits_{-1}^{1} \int\limits_{1}^{2} z^2 y \left[e^x \right]_0^1 \, dy dz = (e-1) \int\limits_{-1}^{1} \int\limits_{1}^{2} z^2 y \, dy dz$$

$$= (e-1) \int\limits_{-1}^{1} z^2 [y^2/2] \Big|_1^2 \, dz$$

$$= \frac{3}{2}(e-1) \int\limits_{-1}^{1} z^2 \, dz = e - 1$$

3.4.2. Integral Over General Regions

We restrict our attention to continuous functions f and to certain simple types of regions.

3 types of region:

Type I – integrating over simple xy-solid

Type II – integrating over simple xz-solid

Type III – integrating over simple yz-solid

Definition 3.5

A solid region G is said to be of **Type 1** if it lies between the graphs of two continuous functions of x and y,

$$G = \{(x, y, z) : x, y \in R, k_1(x, y) \leq z \leq k_2(x, y)\}$$

where R is the projection of G onto the xy-plane, then

$$\iiint_G f(x, y, z)\,dV = \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\,dz \right] dA$$

Type I Regions

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\, dz \right] dA$$

Type II Regions

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x, y, z)\, dy \right] dA$$

Type III Regions

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{h_1(y,z)}^{h_2(y,z)} f(x, y, z)\, dx \right] dA$$
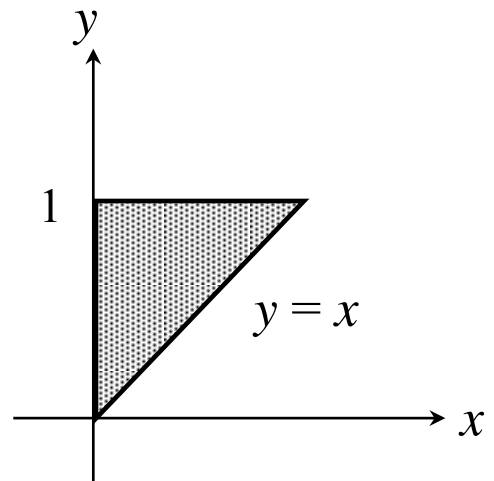
Example 3.16

Let G be the wedge in the first octant cut from the cylindrical solid $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\, dV$$

Solution

♦ **Sketch the solid:** choose Type I
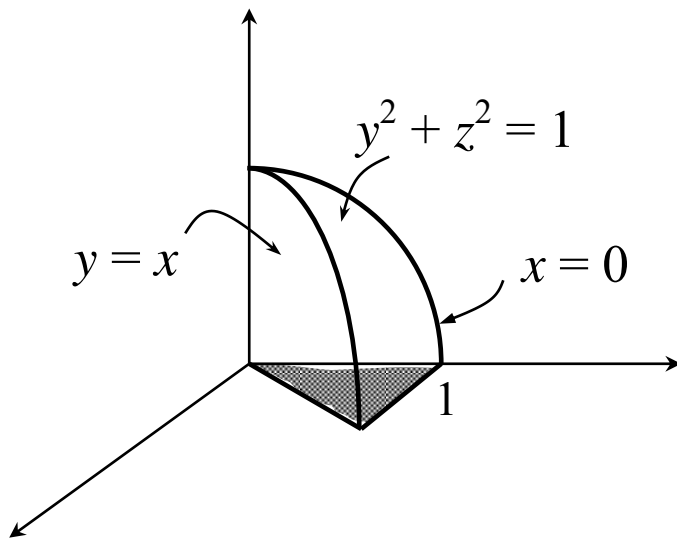
upper bounding surface: $y^2 + z^2 = 1$

lower bounding surface: xy-plane

♦ The z-limits of integration: Draw a line L parallel to z-axis passing through solid region.

As z increases, L enters G at $z = 0$ and leaves at $z = \sqrt{1 - y^2}$

$$\iiint\limits_{G} z\, dV = \iint\limits_{R} \int_{0}^{\sqrt{1-y^2}} \left[z\, dz \right] dA$$

◆ The *x*-limits of integration: Draw a line *M* parallel to *x*-axis passing through plane region *R*.

As *x* increases, *M* enters *R* at $x = 0$ and leaves at $x = y$.

◆ The *y*-limits of integration: Choose *y*-limits that include all lines parallel to the *x*-axis.

The integral is

$$\iiint\limits_{G} z\, dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\, dz\, dx\, dy$$

$$= \int_0^1 \int_0^y \frac{z^2}{2}\Bigg|_0^{\sqrt{1-y^2}} dx\, dy = \int_0^1 \int_0^y \frac{1}{2}(1 - y^2)\, dx\, dy$$

$$= \int_0^1 (1 - y^2) x \Big|_0^y dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\, dy = \frac{1}{8}$$

Alternatively, we evaluate the integral by integrating first with respect to x (Type III).

The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $y = x$.

$$\iiint_G z\,dV = \iint_R \int_0^y \left[z\,dx \right] dA$$

$$y^2 + z^2 = 1$$
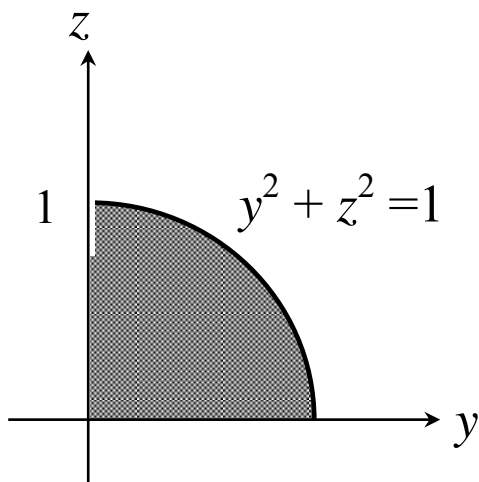
$$\iiint_G z\,dV = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^y z\,dx\,dy\,dz$$

Example 3.17

The volume of a closed bounded region G in space is given by

$$\iiint\limits_{G} dV = \int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx$$

Rewrite the integral as an equivalent iterated integral in the order

(a) $dy\, dz\, dx$ \qquad\qquad (b) $dx\, dy\, dz$

Example 3.18

Find the volume of the region in the first octant bounded by the coordinate planes, the plane $y + z = 2$ and the cylinder $x = 4 - y^2$.

Example 3.19

Find the volume of the region bounded above by $z = 4 - x^2 - y^2$, below by $z = 0$ and laterally by $x^2 + y^2 \leq 1$.

Solution

$$V = \iiint\limits_G dV = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{4-x^2-y^2} dz\,dy\,dx$$

$$= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} 4 - x^2 - y^2 \, dy\,dx$$

$$= 4 \int_0^1 4\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{(\sqrt{1-x^2})^3}{3}\,dx$$

$$\vdots$$

$$= 4 \int_0^{\pi/2} \left(4\cos^2\theta - \sin^2\theta\cos^2\theta - \frac{\cos^4\theta}{3} \right)\theta\,d\theta$$

$$= \cdots = \frac{7}{2}$$

3.4.3 Cylindrical Coordinates

♦ Generalization of polar coordinates in $\mathbb{R}^3$

♦ We convert a triple integral from rectangular to cylindrical coordinates by writing

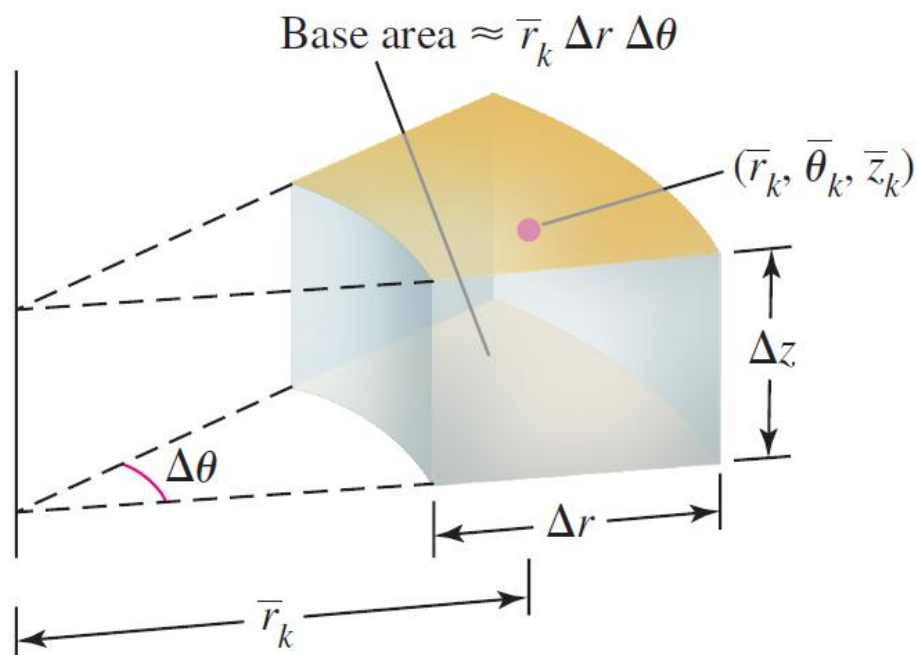$$x = r\cos\theta, \ y = r\sin\theta, \ z = z$$

The element of integration,

$$dV = r\,dr\,d\theta\,dz$$

The function $f(x, y, z)$ is transform to

$$f(x, y, z) = f(r\cos\theta, \ r\sin\theta, \ z)$$

♦ Cylindrical coordinates are convenient for representing cylindrical surfaces and surfaces for which the z-axis is the axis of symmetry.

The cylindrical coordinate system

(x, y, z)
(r, θ, z)

θ

r

(r, θ) or (x, y)

Base area $\approx \bar{r}_k \, \Delta r \, \Delta \theta$

$(\bar{r}_k, \bar{\theta}_k, \bar{z}_k)$

Δz

$\Delta \theta$

Δr

$\bar{r}_k$
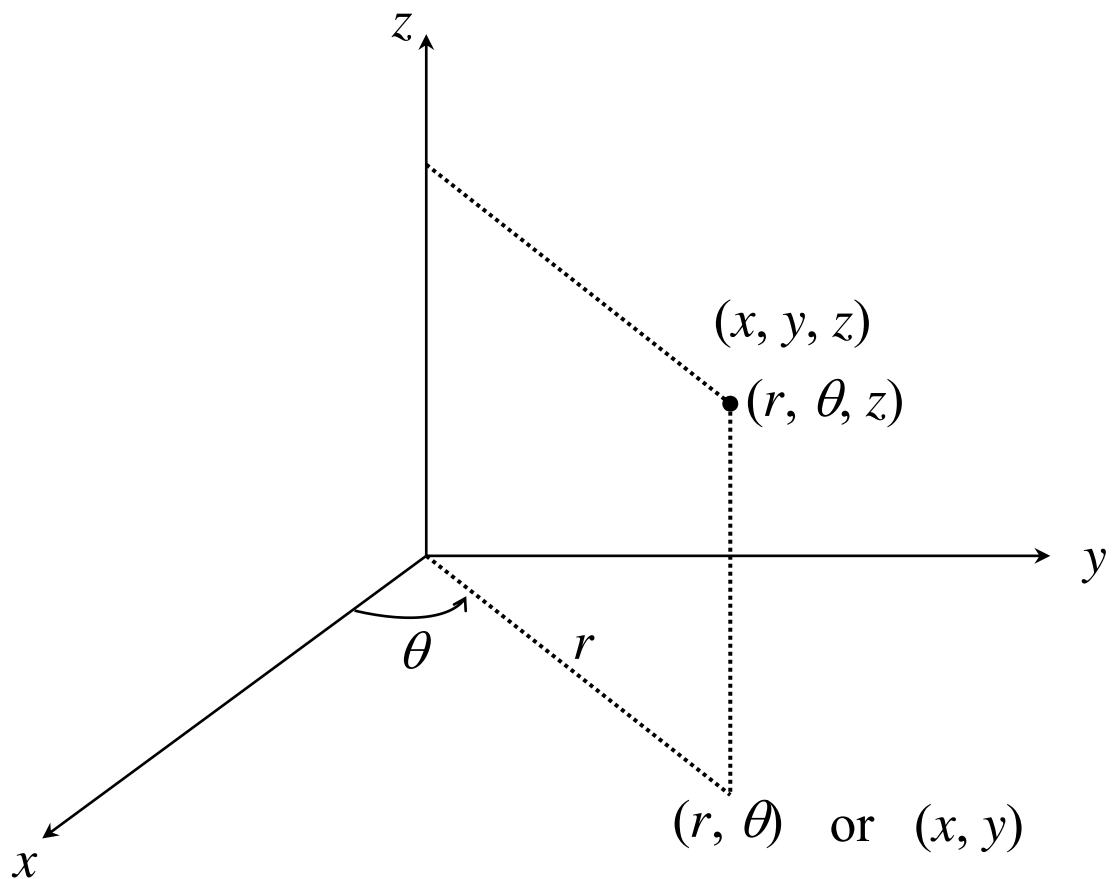
Approximate volume $\Delta V_k \approx \bar{r}_k \, \Delta r \, \Delta \theta \, \Delta z$

Theorem 3.4

Let G be a solid with upper surface $z = g_2(r, \theta)$ and lower surface $z = g_1(r, \theta)$ and let R be the projection of the solid on the xy-plane expressed in polar coordinates. Then if $f(r, \theta, z)$ is continuous on R, we have

$$\iiint\limits_{G} f(r, \theta, z)\, dV = \iint\limits_{R} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, r\, dz\, dr\, d\theta$$

Example 3.20

Use cylindrical coordinates to evaluate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz \, dy \, dx$$

Solution

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz \, dy \, dx = \iiint_{G} x^2 \, dV$$

$$= \int_{0}^{2\pi} \int_{0}^{3} \int_{0}^{9-r^2} r^2 \cos^2 \theta \; r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{3} r^3 \cos^2 \theta z \Big|_{0}^{9-r^2} \, dr \, d\theta$$

$$\vdots$$

$$= \frac{243}{4} \int_{0}^{2\pi} \cos^2 \theta \, d\theta$$

$$= \frac{243}{4} \int_{0}^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{243}{4} \pi$$

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Finding limits in cylindrical coordinates

Example 3.20a

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region G bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + y^2 = 2y$ and above by the paraboloid $z = x^2 + y^2$.

Example 3.21 (Example 3.19 revisited)

Find the volume of the region bounded above by $z = 4 - x^2 - y^2$, below by $z = 0$ and laterally by $x^2 + y^2 \leq 1$.

Example 3.22

Find the volume of solid in the first octant that is bounded by the cylinder $x^2 + y^2 = 2y$, by the cone $z = \sqrt{x^2 + y^2}$ and the xy-plane.
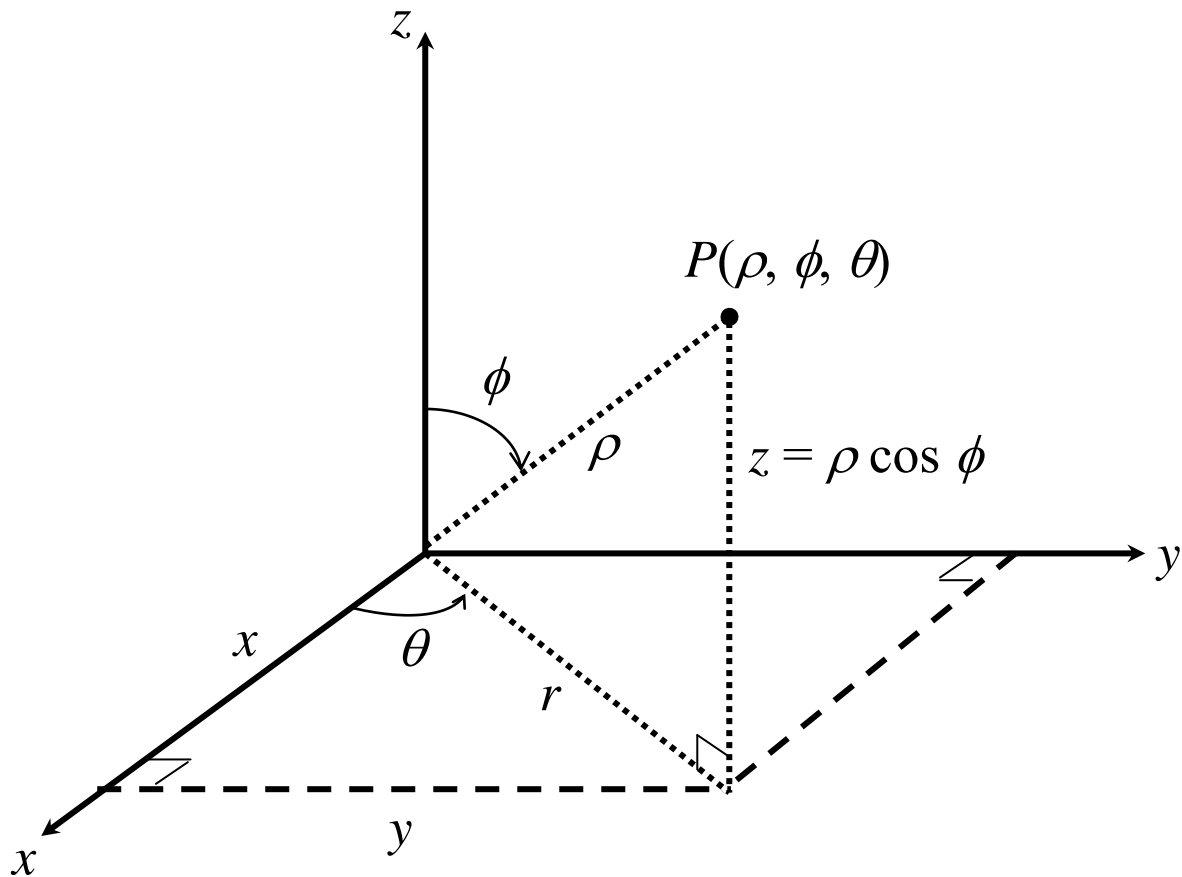
3.4.4 Spherical Coordinates

◆ Useful when you have spherical, or ice-cream cone like surfaces.

◆ Locate points in space with angles and a distance

Definition 3.6

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin

2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$

3. θ is the angle from cylindrical coordinates.
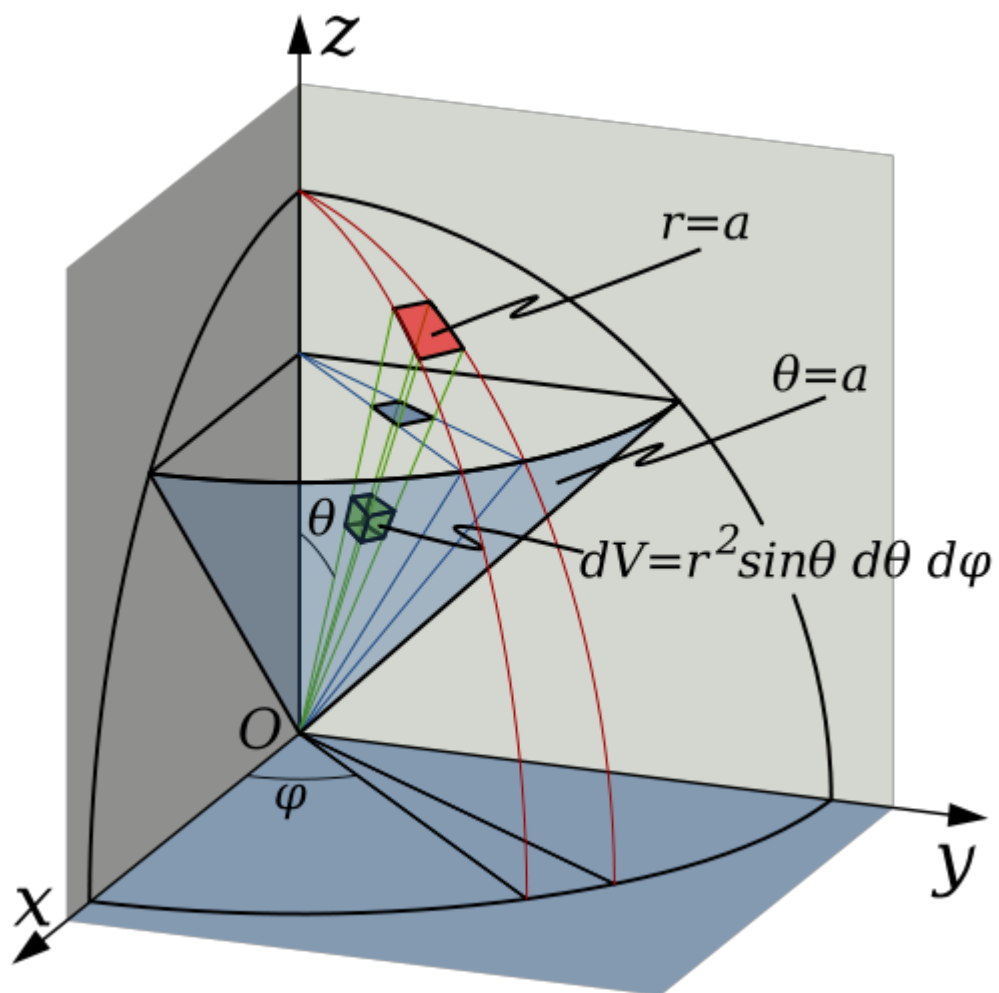
The spherical coordinate system
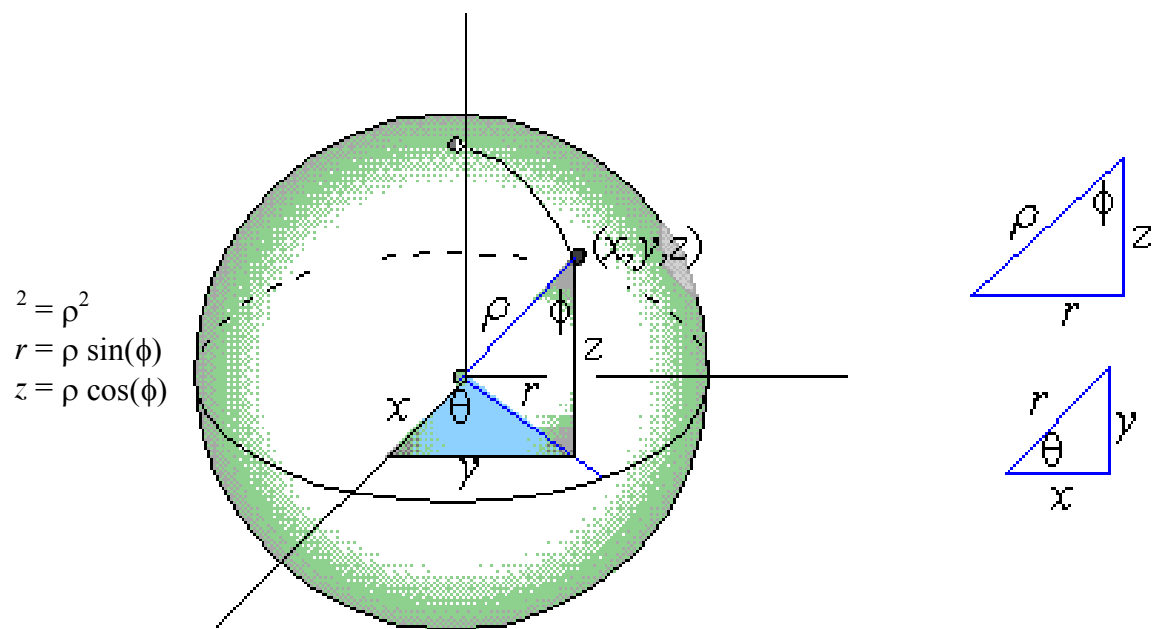

Since $r = \rho \sin \phi$,

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

and $z = \rho \cos \phi$, $\quad x^2 + y^2 + z^2 = \rho^2$

$$^2 = \rho^2$$
$$r = \rho \sin(\phi)$$
$$z = \rho \cos(\phi)$$

$r = a$

$\theta = a$

$dV = r^2 \sin\theta \, d\theta \, d\varphi$

Integrate first in ρ
with $g(\varphi, \theta) \le \rho \le h(\varphi, \theta);...$

... then integrate in φ and θ
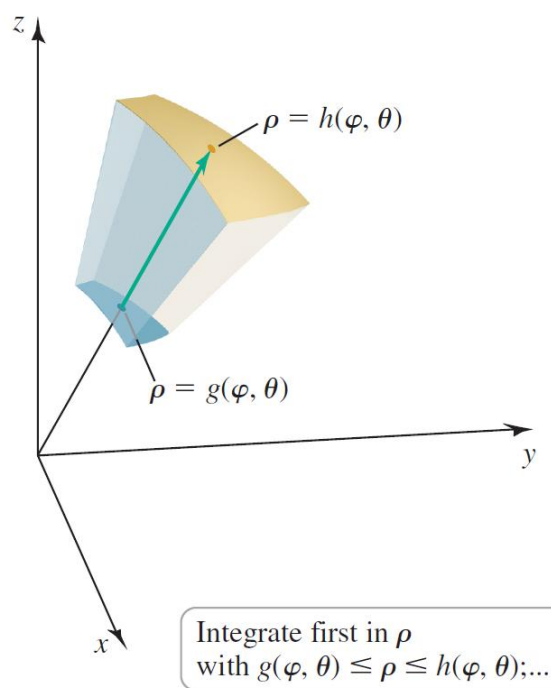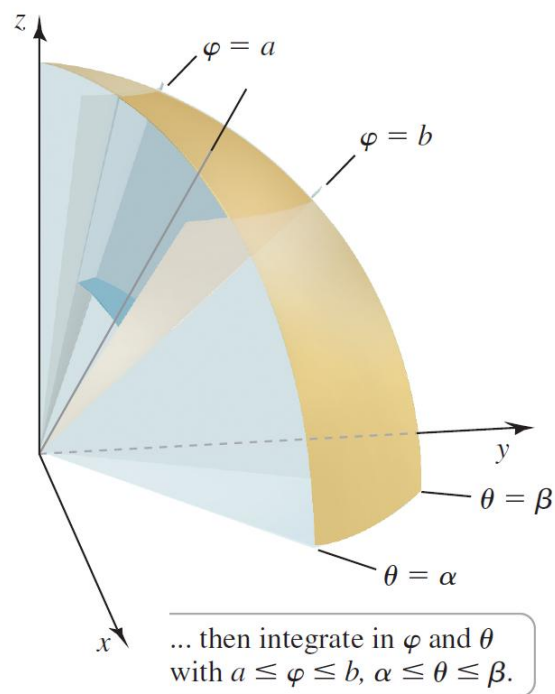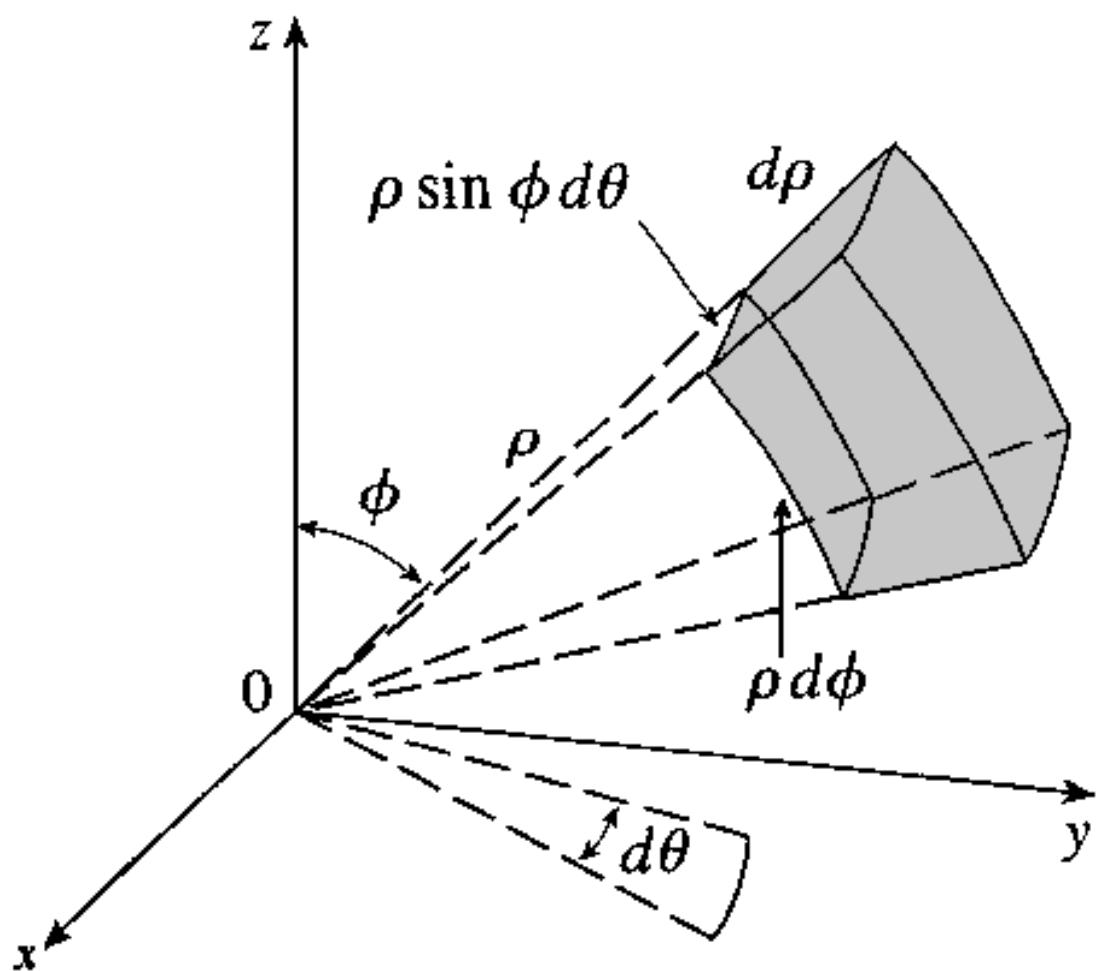with $a \le \varphi \le b, \alpha \le \theta \le \beta.$

FIGURE 13.59

♦ The function $f(x, y, z)$ is transform to

$f(x, y, z) =$
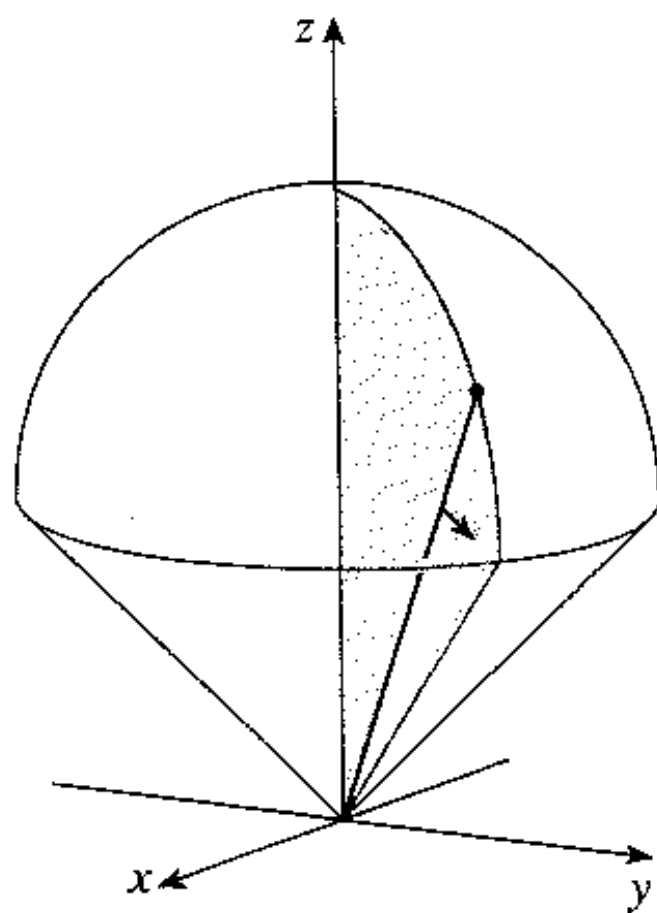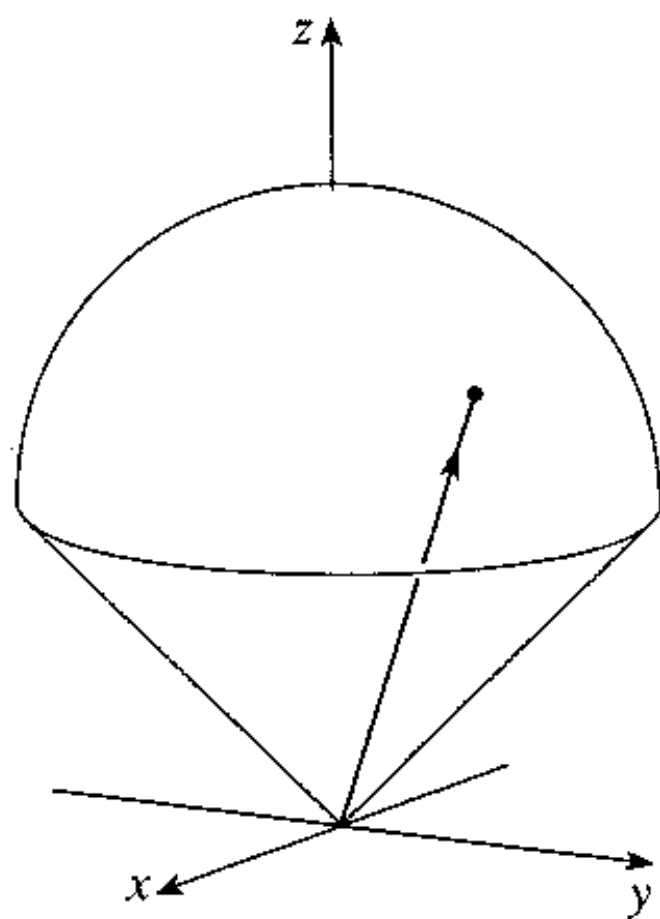 $f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$
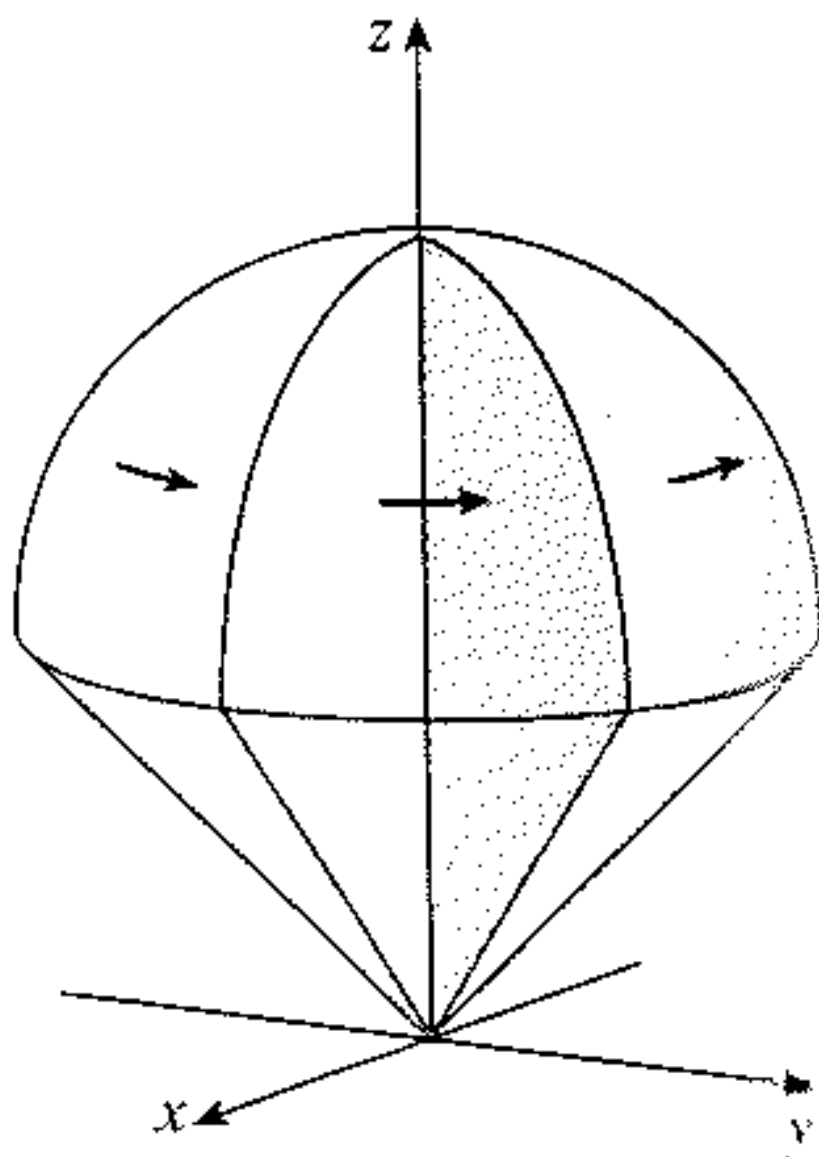
♦ The element of integration,

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

♦ Triple integrals in spherical coordinates are then evaluated as iterated integrals. The integral is

$$\iiint\limits_G f(\rho, \phi, \theta) \, dV = \iiint\limits_G f(\rho, \phi, \theta) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Example 3.23

Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx$$

Solution

Sketch the region G of integration.

♦   From the z-limits of integration:

The upper surface of G is the hemisphere $z = \sqrt{4 - x^2 + y^2}$ and the lower surface is the xy-plane $z = 0$.

♦   From the x- and y-limits of integration:

The projection of the solid G on the xy-plane is the region enclosed by the circle $x^2 + y^2 = 4$.
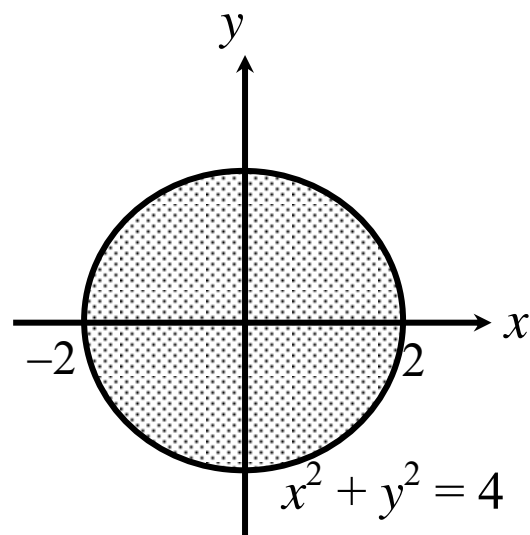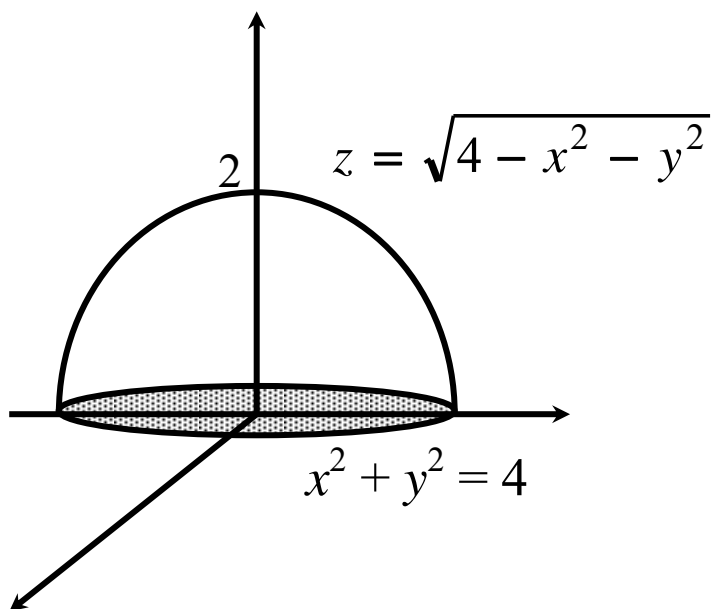
$$\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2 + y^2 + z^2}\ dz\,dy\,dx$$

$$= \iiint\limits_{G} z^2 \sqrt{x^2 + y^2 + z^2}\ dV$$

$$= \int\limits_{0}^{2\pi} \int\limits_{0}^{\pi/2} \int\limits_{0}^{2} \rho^5 \cos^2 \phi \sin \phi\ d\rho\,d\phi\,d\theta$$

$$= \int\limits_{0}^{2\pi} \int\limits_{0}^{\pi/2} \cos^2 \phi \sin \phi\ \left.\frac{\rho^6}{6}\right|_{0}^{2}\ d\phi\,d\theta$$

$$= \frac{32}{3} \int\limits_{0}^{2\pi} \int\limits_{0}^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \, d\theta$$

$$= \frac{32}{3} \int\limits_{0}^{2\pi} \left. -\frac{\cos^3 \phi}{3} \right|_{0}^{\pi/2} d\theta$$

$$= \frac{32}{3} \int\limits_{0}^{2\pi} d\theta = \frac{64}{9} \pi$$

Example 3.24

Let G be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of G using the following orders of integration.

(a) $d\rho \, d\phi \, d\theta$ (b) $d\phi \, d\rho \, d\theta$

Example 3.24a

Let G be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$ and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in spherical coordinates that give the volume of G.

Example 3.25

Find the volume of the "ice cream cone" G cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \dfrac{\pi}{3}$.

Summary

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$r = \rho\sin\phi$ $z = \rho\cos\phi$ $\theta = \theta$

Corresponding volume elements

$$dV = dz\,dy\,dx$$

$$= dz\,r\,dr\,d\theta$$

$$= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

3.5 Moments and Centre of Mass

3.5.1 Notation and Terminology

Lamina - a solid object that is sufficiently "flat" to be regarded as two-dimensional.

Density: mass per unit area, $\delta(x, y)$

Mass: quantity of matter in a body, m

Moment of mass: tendency of mass to produce a rotation about a point, line or plane

   Positive moment – clockwise rotation

   Negative moment – counterclockwise rotation

Center of Gravity/Center of Mass:

a point where a system behaves as if all its mass is concentrated there (balance point).

Centroid: center of mass of a homogeneous body

Moment of inertia: tendency to resist a change in the rotational motion about an axis.

Definition 3.6

If δ is a continuous density function on the lamina corresponding to a plane region R, then

♦ Mass, $m = \iint\limits_R \delta(x, y)\, dA$

♦ Moments of mass about the x- and y-axes,

$$M_x = \iint\limits_R y\, \delta(x, y)\, dA$$

$$M_y = \iint\limits_R x\, \delta(x, y)\, dA$$

♦ Centre of mass

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

♦ If the density δ is constant, the point $(\bar{x}, \bar{y})$ is called the centroid of the region.

Example 3.26

A lamina of density $\delta(x, y) = x^2$ occupies a region R bounded by the parabola $y = 2 - x^2$ and the line $y = x$. Find

(a) mass

(b) centre of mass

of the lamina.

Solution

♦ sketch the region R

(a) mass of lamina,

$$m = \iint\limits_{R} \delta(x, y) \, dA = \int\limits_{-2}^{1} \int\limits_{x}^{2-x^2} x^2 \, dy \, dx$$

$$= \int\limits_{-2}^{1} x^2 \, y \Big|_{x}^{2-x^2} \, dx$$

$$\therefore \; m = \int\limits_{-2}^{1} (2x^2 - x^4 - x^3) \, dx = \frac{63}{20}$$

(b) centre of mass, $(\bar{x}, \bar{y})$

KNOW: $\bar{x} = \dfrac{M_y}{m}, \quad \bar{y} = \dfrac{M_x}{m}$

$$M_x = \iint\limits_{R} y \, \delta(x, y) \, dA$$

$$= \int\limits_{-2}^{1} \int\limits_{x}^{2-x^2} y \, x^2 \, dy \, dx = \int\limits_{-2}^{1} x^2 \, \frac{y^2}{2} \Big|_{x}^{2-x^2} \, dx$$

4

$$\therefore M_x = \frac{1}{2} \int_{-2}^{1} (x^6 - 5x^4 + 4x^2) \, dx = -\frac{9}{7}$$

$$M_y = \iint_R x \, \delta(x, y) \, dA$$

$$= \int_{-2}^{1} \int_{x}^{2-x^2} x^3 \, dy \, dx = \int_{-2}^{1} x^3 \, y \Big|_{x}^{2-x^2} \, dx$$

$$\therefore M_y = \int_{-2}^{1} (2x^3 - x^5 - x^4) \, dx = -\frac{18}{5}$$

From (a) we found $m = \frac{63}{20}$, so the centre of mass is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{-18/5}{63/20} = -\frac{8}{7} \approx -1.14$$

$$\bar{y} = \frac{M_x}{m} = \frac{-9/7}{63/20} = -\frac{20}{49} \approx -0.41$$

In an analogous way, we can use the triple integral to find mass and the center of mass of a solid in $\mathbb{R}^3$. The density $\delta(x, y, z)$ at a point in the solid now refers to mass per unit volume.

♦ Mass $\quad m = \iiint\limits_{G} \delta(x, y, z)\, dV$

♦ Moments $\quad M_{yz} = \iiint\limits_{G} x\, \delta(x, y, z)\, dV$

$$M_{xz} = \iiint\limits_{G} y\, \delta(x, y, z)\, dV$$

$$M_{xy} = \iiint\limits_{G} z\, \delta(x, y, z)\, dV$$
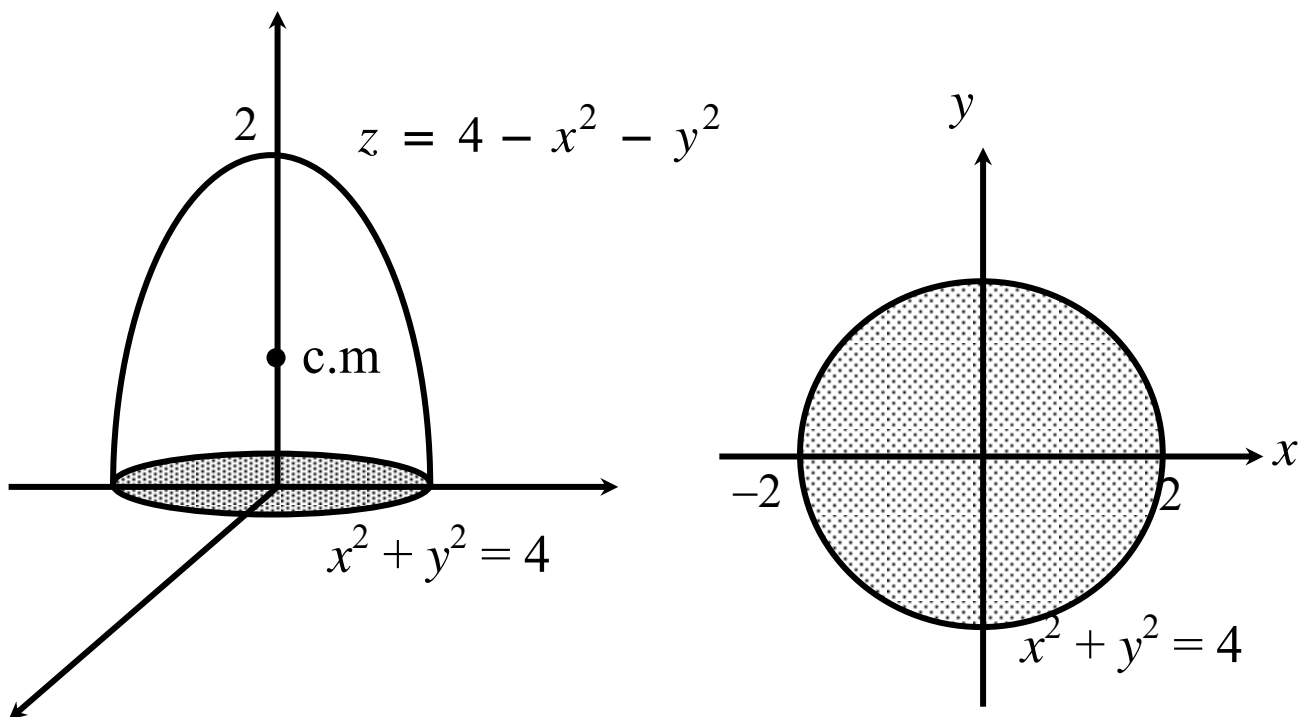
♦ Centre of mass

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

♦ If the density δ is constant, the point $(\bar{x}, \bar{y}, \bar{z})$ is called the centroid.

Example 3.27

Find the centroid of a solid of constant density δ bounded below by the disk $x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

Solution

By symmetry, $\bar{x} = \bar{y} = 0$. So we only need to find $\bar{z}$.

$$\bar{z} = \frac{M_{xy}}{m}$$

$$M_{xy} = \int\int_G\int z\, \delta(x,\, y,\, z)\, dV$$

$$= \int\int_R \int_0^{4-x^2-y^2} z\delta\, dz\, dy\, dx$$

$$= \int\int_R \delta\, \frac{z^2}{2}\bigg|_0^{4-x^2-y^2} dy\, dx$$

$$= \frac{\delta}{2} \int\int_R (4 - x^2 - y^2)^2\, dy\, dx$$

$$= \frac{\delta}{2} \int_0^{2\pi}\int_0^2 (4 - r^2)^2 r\, dr\, d\theta$$

$$= \frac{\delta}{2} \int_0^{2\pi} -\frac{1}{6}(4 - r^2)^3\bigg|_0^2 dx$$

$$= \frac{16\delta}{3} \int_0^{2\pi} d\theta$$

$$\therefore\ M_{xy} = \frac{32\pi\delta}{3}$$

A similar calculation gives

$$m = \iiint_G \delta(x, y, z)\, dV$$

$$= \iint_R \int_0^{4-x^2-y^2} \delta\, dz\, dy\, dx \;=\; 8\pi\delta$$

Therefore $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{32\pi\delta/3}{8\pi\delta} = \dfrac{4}{3}$.

Thus the centroid is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.

Example 3.28

A solid is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 2$. If the density $\delta(x, y, z) = 2x$, find the centre of mass.

3.5.3 Moments of Inertia
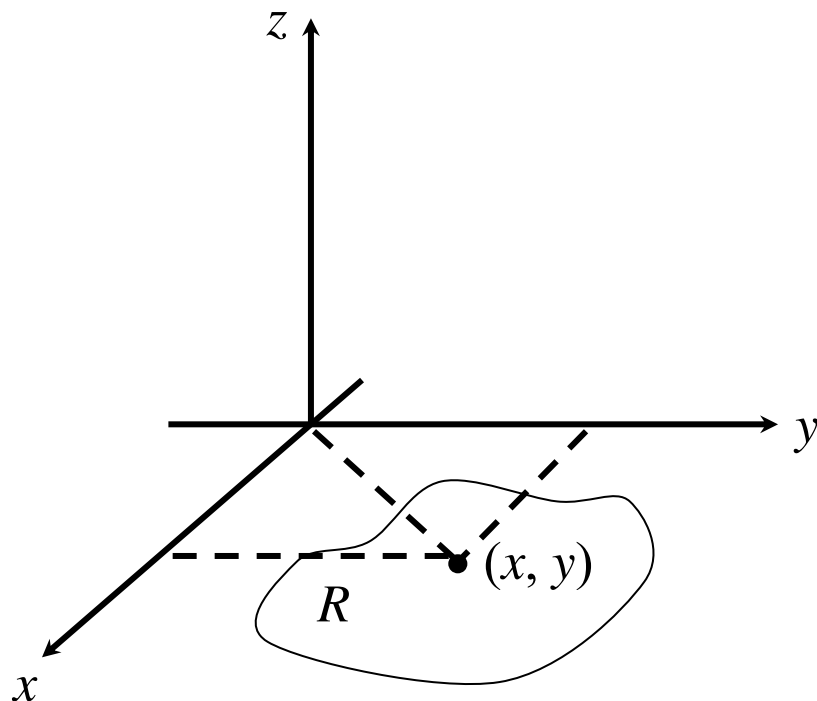
◆ Also called the second moments

Definition 3.7

The moments of inertia of a lamina of density δ covering the planar region R about the x-, y-, and z-axis are given by

$$I_x = \int\int_R y^2 \delta(x, y) \, dA$$

$$I_y = \int\int_R x^2 \delta(x, y) \, dA$$

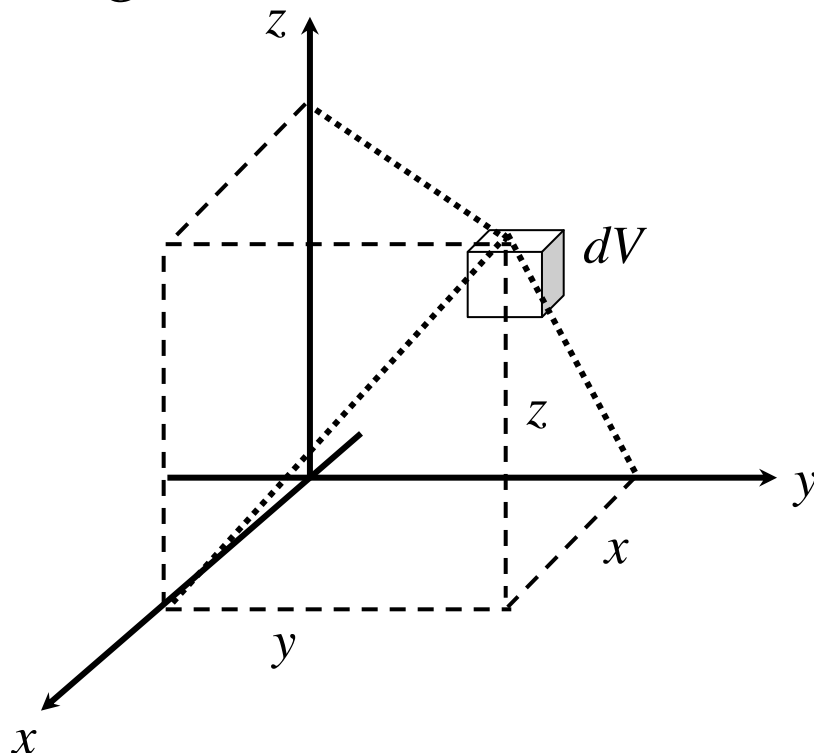$$I_z = \int\int_R (x^2 + y^2)\delta(x, y) \, dA$$

The concept of moments of inertia generalise easily to solid regions.

Suppose the solid occupies a region R and that the density at each point (x, y, z) in R is given by $\delta(x, y, z)$. The moments of inertia of the solid about the x-, y-, and z-axis are given by

$$I_x = \iiint\limits_{G} (y^2 + z^2)\delta(x, y, z)\, dV$$

$$I_y = \iiint\limits_{G} (x^2 + z^2)\delta(x, y, z)\, dV$$

$$I_z = \iiint\limits_{G} (x^2 + y^2)\delta(x, y, z)\, dV$$

Example 3.29

A lamina of density $\delta(x, y) = x^2 y$ occupies the region R in the plane that is bounded by the parabola $y = x^2$ and the lines $x = 2$ and $y = 1$. Find the moments of inertia of the lamina about the x-axis and the y-axis.

Example 3.30

Find the moment of inertia of the "ice cream cone" G cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \dfrac{\pi}{3}$ about the z-axis. (Take $\delta = 1$)

Example 3.31

Find the moment of inertia of a solid hemisphere of radius 2 with respect to its axis of symmetry, if the density is proportional to the distance from the axis of symmetry.