CHAPTER 1: Infinite Sequences

1.1 Definition of Infinite Sequences

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1.3 Convergent and Divergent Sequence

1.4 The Sandwich Theorem

1.1 Definition of Infinite Sequences

A sequence is nothing more than a list of numbers written in a specific order. General terms:

$$
\begin{array}{lcl}
a_1 & - & \text{1st term} \\
a_2 & - & \text{2nd term} \\
\vdots & & \\
a_n & - & n\text{th term} \\
a_{n+1} & - & (n+1)\text{ term} \\
\vdots & &
\end{array}
$$

This notation tells us that the sequences continue on and does not terminate at the last term. It shows an infinite sequence.

$a_{n+1} \neq a_n + 1$ Be careful with this notation.

There is a variety ways of denoting a sequence:

$$\{a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots\} \text{ or } \{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

a_n is usually given a formula.

Example

Write down the first 5 terms of the following sequences.

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{-1^{\,n+1}}{2^n}\right\}_{n=0}^{\infty}$

Solution

(a) To get the first 5 terms here all we need to do is plug in values of *n* into the formula given and we will get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \quad , \quad , \quad , \quad , \quad , \cdots$$

(b) This one is similar to example (a). The main different is that this sequence does not start at n=1.

$$\left\{\frac{-1^{\,n+1}}{2^n}\right\}_{n=0}^{\infty} = \quad , \quad , \quad , \quad , \quad , \cdots$$

The terms in this sequence alternate in signs = alternating sequences. In these examples, we were really treating the formula as functions that can only have integers plugged into them:

$$f(n) = \frac{n+1}{n^2}, g(n) = \frac{(-1)^{n+1}}{2^n}.$$

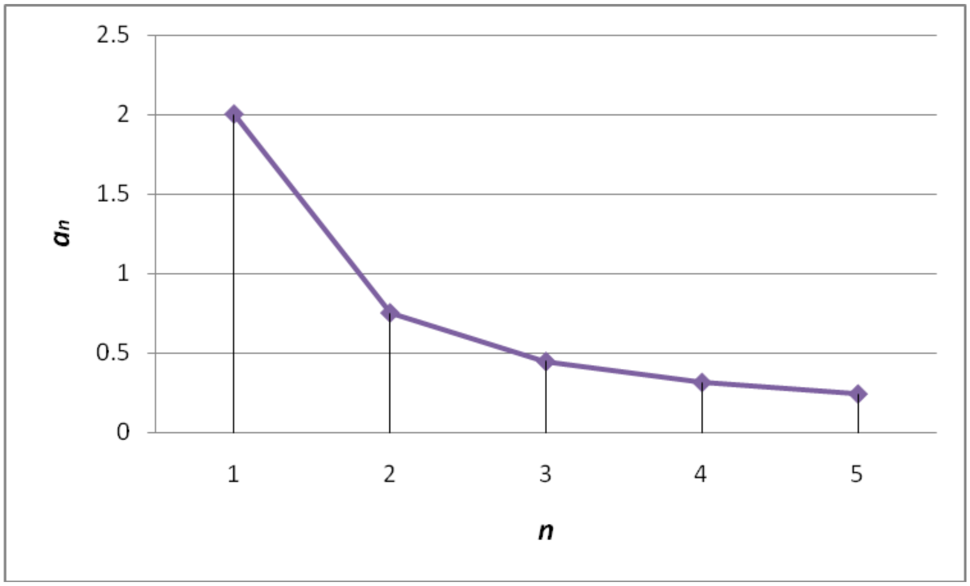
Notes: Treating the sequence terms as function evaluations will allow us to do many things with sequences.

First we want to think about "graphing" a sequence. To graph the sequence $\{a_n\}$, we plot the points (n, a_n) as n ranges over all possible values on a graph.

For instance, let's graph the sequence $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$ such as given in example (a) above.

First few points on the graph are

$$(1,2),\left(2,\frac{3}{4}\right),\left(3,\frac{4}{9}\right),\left(4,\frac{5}{16}\right),\left(5,\frac{6}{25}\right),\cdots$$



The graph leads us to an important idea about sequences. Notice that *n* increases, the sequence terms in our sequence in this case, get closer and closer to zero.

We then say that zero is the limit of the sequence and write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0.$$

The same notation we use when we talked about the limit of a function.

1.2 Techniques for Finding Limits

$$\lim_{n\to\infty} a_n = L$$

The value of a_n's approach *L* as *n* approaches infinity.

$$\lim_{n\to\infty} a_n = \infty$$

The value of a_n's get larger and larger without bound as *n* approaches infinity.

$$\lim_{n\to\infty} a_n = -\infty$$

The value of a_n's negative and get larger and larger without bound as *n* approaches infinity.

Suppose that $\left\{n^{\alpha}\right\}(\alpha > 0), \left\{e^{n}\right\}, \{\ln n\}, \{\sin n\}$

and $\left\{\cos n\right\}$ are sequences. Then

$$\lim_{n\to\infty} n^{\alpha} = \infty,$$

$$\lim_{n\to\infty} e^n = \infty,$$

$$\lim_{n\to\infty} \ln n = \infty,$$

$$\lim_{n\to\infty} \sin n = \lim_{n\to\infty} \cos n = \text{does not exist}$$
$$(\neq \text{real no. and } \neq \pm\infty)$$

Properties of Limits

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences. Suppose that $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$ and $\lim_{n\to\infty} c_n = \infty$ with *L* and *M* are real numbers. Then

1) $\lim_{n\to\infty}\left[a_n \pm b_n\right] = L \pm M$

2) $\lim_{n\to\infty}\left[a_n b_n\right] = LM$

3) $\lim_{n\to\infty}\left[\dfrac{a_n}{b_n}\right] = \dfrac{L}{M}, \quad M \neq 0$

4) $\lim_{n\to\infty}\left[c a_n\right] = cL$, *c* is a constant.

5) $\lim_{n\to\infty} f\left(a_n\right) = f\left(\lim_{n\to\infty} a_n\right)$ f is continuous.

6) $\lim_{n\to\infty} \left|a_n\right| = \left|\lim_{n\to\infty} a_n\right| = \left|L\right|$

7) $\lim_{n\to\infty} \left[c_n \pm b_n\right] = \infty$

8) $\lim_{n\to\infty} \left[c_n b_n\right] = \infty \quad M > 0$

9) $\lim_{n\to\infty} \left[\frac{c_n}{b_n}\right] = \infty, \quad M > 0$

10) $\lim_{n\to\infty} \left[c_n b_n\right] = \infty \quad M < 0$

11) $\lim_{n\to\infty} \left[\frac{c_n}{b_n}\right] = \infty, \quad M < 0$

12) $\lim_{n\to\infty} \left[\frac{b_n}{c_n}\right] = 0$

1.3 Convergent and Divergent Sequence

If $\lim\limits_{n\to\infty} a_n$ exists and is finite, we say that the sequence is convergent. If $\lim\limits_{n\to\infty} a_n$ does not exists or is infinite, we say that the sequence is divergent.

Example

Determine if each sequence converges or diverges; if it converges state its limit.

(a) $\left\{ \dfrac{5}{e^n} \right\}$

(b) $2 + \ln n$

(c) $\left\{ -1^{\;n} \right\}$

(d) $\left\{ \sin\left(\dfrac{1}{n} \right) \right\}$

(e) $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}$

L'Hopital's Rule

When using the limit properties, we may encounter indeterminate form 0/0 and ∞/∞. Here we use L'Hopital rule. Suppose that *f* and *g* are differentiable and

$$\lim_{n\to\infty} f(n) = 0 = \lim_{n\to\infty} g(n) \text{ or}$$
$$\lim_{n\to\infty} f(n) = \infty = \lim_{n\to\infty} g(n).$$

Then, $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = \lim_{n\to\infty} \dfrac{f'(n)}{g'(n)}$.

Example

Find the limit of each of the following sequence.

(a) $\left\{\left(\frac{n}{2n+1}\right)^3\right\}$

(b) $\left\{\frac{7-4n^2}{3+5n^2}\right\}$

Other indeterminate case that we may encounter when we use limit properties or theorems are $(\infty)(0)$, 0^0, 1^∞, ∞^0 and $(\infty - \infty)$.

For these cases, we have to do something with the expressions of the sequences, such as taking logarithms, rearranging or combining the terms so that we can apply L'Hopital's rule.

Example

Find the limit of each of the following sequence.

(a) $\left\{ n \ln \left(1 + \frac{1}{n} \right) \right\}$

(b) $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$

1.4 The Sandwich Theorem

Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences and for every integer $n \geq 1$, we have

$$a_n \leq b_n \leq c_n.$$

If $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$ then $\lim\limits_{n\to\infty} b_n = L$.

Example

Find the limit of each of the following sequence.

(a) $\left\{\dfrac{1}{n\,!}\right\}$

(b) $\left\{\dfrac{\cos^2 n}{3^n}\right\}$

(c) $\left\{\dfrac{\cos \pi n}{n^2}\right\}$

(d) $\left\{\dfrac{\sin^2 n}{n\,!}\right\}$