CHAPTER 2: Infinite Series

2.1 Definition of Infinite Series

$\{a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots\}$ - infinite sequence

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_k = a_1 + a_2 + a_3 + \cdots + a_k = \sum_{n=1}^{k} a_n$$

The sequence of partial sum, $\{S_k\}$ = Infinite series or for short series only.

If there is a real no. S such that $\lim_{k\to\infty} S_k = S$,

that is $\sum_{n=1}^{\infty} a_n = S$.

= the sum of the series

Then we say that the series converges,

$\sum_{n=1}^{\infty} a_n$ converges.

If $\lim_{k\to\infty} S_k$ does not exist or $\lim_{k\to\infty} S_k = \pm\infty$,

Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

2.2 Telescoping and Geometric Series

2.2.1 Telescoping Series

A series $\sum_{n=1}^{\infty} a_n$ is a telescoping series if there is a sequence $\{b_n\}$ such that

$$a_n = b_n - b_{n+1} \quad ; \quad n = 1, 2, 3, \ldots$$

Then $\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$. Hence the telescoping series converges if and only if the sequence $\{b_n\}$ converges.

Example

Show that the following series is a telescoping series. Hence, determine the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \quad n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n+3}\right)$

Example

Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \quad n+2}$.

2.2.2 Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \quad ; \quad a \neq 0$$

Note: *a* is the first term.

The geometric series is convergent if $|r| < 1$ and its sum is $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$. If $|r| > 1$, then the geometric series is divergent.

Example

(a) Find the sum of geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

(b) Is the series $\sum_{n=0}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

2.3 The Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0, n = 1, 2, \ldots$ and f be a function such that $f(n) = a_n$. f is continuous and decreasing function for all real $x \geq 1$ and *L* is a real number.

(1) If $\int_1^{\infty} f(x)\, dx = L$ then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\int_1^{\infty} f(x)\, dx = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Notes: *This test cannot be used to calculate sum. *Use this test when $f(x)$ is easy to integrate. *This test only applies to series that have positive terms.

Example

Determine whether the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n+5}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note: The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called *p*-series. The *p*-series converges if $p > 1$ and diverges if $0 < p \leq 1$.

2.4 Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may be convergent or may be divergent, hence other tests should be used.

Note: This test only determines the divergence of a series.

Example

Show that the series diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \cos n\pi$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{{n+1/n}^{\,n}}$

2.5 Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms such that

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3, \ldots, a_n \le b_n, \ldots$$

for all *n*. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Notes:

*The series that we use for comparison are usually the geometric series or the p-series.

*Use this test as a last resort; other tests are often easier to apply.

*This test only applies to series with nonnegative terms.

Example

Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^2 + 5}$

2.5.1 Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms such that $c = \lim_{n \to \pm\infty} \frac{a_n}{b_n}$. If $0 < c < \infty$, then both series converge or both diverge.

Note: This is easier to apply than the comparison test, but still requires some skill in choosing the series $\sum_{n=1}^{\infty} b_n$ for comparison.

If $c = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $c = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example

Determine whether each series converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n - n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}$