

**DEPARTMENT OF MATHEMATICS**  
**FACULTY OF SCIENCE**  
**UNIVERSITI TEKNOLOGI MALAYSIA**

**SSH 1033 MATHEMATICAL METHODS II**

**TUTORIAL 2**

1. Find the first four partial sums; find a closed form for the  $n$ th partial sum; and determine whether the series converges (if so, give the sum).

$$(a) \sum_{k=1}^{\infty} \frac{2}{5^{k-1}}.$$

$$(b) \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}.$$

2. Determine whether the series converges or diverges. If it converges, find the sum.

$$(a) \sum_{k=1}^{\infty} \frac{1}{5^k}.$$

$$(b) \sum_{k=1}^{\infty} \left(-\frac{3}{4}\right)^{k-1}.$$

$$(c) \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2}.$$

$$(d) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{7}{6^{k-1}}.$$

$$(e) \sum_{k=1}^{\infty} 4^{k-1}.$$

$$(f) \sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^{k+1}.$$

$$(g) \sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right).$$

$$(h) \sum_{k=2}^{\infty} \frac{1}{k^2 - 1}.$$

3. Prove that the infinite series (sometimes called the *geometric series*)

$$a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$$

(a) converges to  $\frac{a}{1-r}$  if  $|r| < 1$ .

(b) diverges if  $|r| \geq 1$ .

4. Find the sum of the series (a)  $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$ . (b)  $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{5^k}$ .

5. Determine whether the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges or diverges. If it converges, find the sum.

6. Show that

$$\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{3}{2}.$$

7. Show that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots = \frac{1}{2}.$$

8. Show that

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \cdots = \frac{3}{4}$$

9. Express the following recurring decimals as rational number.

- (a) 0.4444...    (b) 0.9999...    (c) 5.373737...
- (d) 0.159159...    (e) 0.451141414...

10. Find a closed form for the  $n$ th partial sum of the series

$$\ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \cdots + \ln \frac{n}{n+1} + \cdots$$

and determine whether the series converges.

11. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is  $\frac{3}{4}$  of the previous height. Find the total distance the ball will travel if it is allowed to bounce indefinitely.

12. Show the following.

$$(a) \sum_{k=2}^{\infty} \ln \left(1 - \frac{1}{k^2}\right) = -\ln 2. \quad (b) \sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1.$$

13. Use geometric series to show that

$$(a) \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}, \quad \text{if } -1 < x < 1.$$

$$(b) \sum_{k=0}^{\infty} (x-3)^k = \frac{1}{4-x}, \quad \text{if } 2 < x < 4.$$

$$(c) \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}, \quad \text{if } -1 < x < 1.$$

14. Find the values of  $x$  for which the series converges, and for these values find its sum.

- (a)  $x - x^3 + x^5 - x^7 + x^9 - \cdots$
- (b)  $\frac{1}{x^2} + \frac{2}{x^3} + \frac{4}{x^4} + \frac{8}{x^5} + \frac{16}{x^6} + \cdots$
- (c)  $e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \cdots$
- (d)  $\sin x - \frac{1}{2} \sin^2 x + \frac{1}{4} \sin^3 x - \frac{1}{8} \sin^4 x + \cdots$

15. Show that

$$\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \frac{1}{7 \cdot 9 \cdot 11} + \cdots = \frac{1}{12}.$$

16. Find the sum of the following series.

$$(a) \sum_{k=1}^{\infty} \left[ \frac{1}{2^k} + \frac{1}{4^k} \right].$$

$$(c) \sum_{k=2}^{\infty} \left[ \frac{1}{k^2 - 1} - \frac{7}{10^{k-1}} \right].$$

$$(b) \sum_{k=1}^{\infty} \left[ \frac{1}{5^k} - \frac{1}{k(k+1)} \right].$$

$$(d) \sum_{k=1}^{\infty} \left[ \frac{7}{3^k} + \frac{6}{(k+3)(k+4)} \right].$$

17. Use the divergence test to show that the series diverges.

$$(a) \sum_{k=1}^{\infty} \frac{k+1}{k+2}.$$

$$(c) \sum_{k=1}^{\infty} \frac{k^2 + k + 3}{2k^2 + 1}.$$

$$(e) \sum_{k=1}^{\infty} \cos k\pi.$$

$$(b) \sum_{k=1}^{\infty} \ln k.$$

$$(d) \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^k.$$

$$(f) \sum_{k=1}^{\infty} \frac{e^k}{k}.$$

18. Determine whether the following series converges or diverges.

$$(a) \sum_{k=1}^{\infty} \frac{1}{k+6}.$$

$$(c) \sum_{k=1}^{\infty} \frac{1}{5k+2}.$$

$$(e) \sum_{k=1}^{\infty} \frac{1}{1+9k^2}.$$

$$(g) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}.$$

$$(i) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{1/3}}.$$

$$(k) \sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}.$$

$$(m) \sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}.$$

$$(o) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}.$$

$$(b) \sum_{k=1}^{\infty} \frac{3}{5k}.$$

$$(d) \sum_{k=1}^{\infty} \frac{k}{k^2+1}.$$

$$(f) \sum_{k=1}^{\infty} \frac{1}{(2k+4)^{3/2}}.$$

$$(h) \sum_{k=1}^{\infty} \frac{1}{e^{1/k}}.$$

$$(j) \sum_{k=1}^{\infty} \frac{\ln k}{k}.$$

$$(l) \sum_{k=1}^{\infty} ke^{-k^2}.$$

$$(n) \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^{-k}.$$

$$(p) \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}.$$

19. Prove that  $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

20. Prove that  $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

### Answers Tutorial 2

1. (a)  $S_4 = 312/125$ ;  $S_n = 5/2 [1 - (1/5)^n]$  (b)  $S_4 = 1/3$ ;  $S_n = 1/2 - 1/(n+2)$
2. All converge except (e) and (f). (a) 1/4 (b) 4/7 (c) 8/9 (d) 6 (g) 1/2 (h) 3/4
4. (a) -1/3 (b) 1/6
5. converges; 1
9. (a) 4/9 (b) 1 (c) 532/99 (d) 159/999 (e) 44663/99000
10.  $S_n = -\ln(n+1)$ ; diverges
11. 70 m
14. (a)  $\{x : -1 < x < 1\}$  (b)  $\{x : x < -2 \text{ or } x > 2\}$  (c)  $\{x : x > 0\}$   
(d)  $\{x : -\infty < x < \infty\}$
16. (a) 4/3 (b) -3/4 (c) -1/36 (d) 5
18. (a), (b), (c), (d), (g), (h), (i), (j), (k), (m), (n), (o), diverge

### Answers Tutorial 3

1. (a), (d) converge, (b), (e) diverge, (c), (f) inconclusive
2. (c) converges, (a), (b) diverge (d) inconclusive
3. (b), (c), (e), (i), (m), (n), (p) converge; others diverge
8. (a), (c), (f) converge; others diverge
9. (a), (e), (f), (h), (i), (j), (l) converge; others diverge
10. converges
11. diverges
13. all converges except (c)
14. (c), (f) diverge; others converge absolutely
15. (b), (d), (g), (h), (l), (o) converge absolutely; (c), (k) diverge;  
others converge conditionally

**Note :** In your quest for some of the answers you may want to use the result

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \text{ and } \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}.$$