

*\* Omit these exercises.*

DEPARTMENT OF MATHEMATICS  
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SSH 1033 MATHEMATICAL METHODS II

TUTORIAL 1

1. Write the first five terms of each of the following sequences.

(a)  $\left\{ \frac{2n-1}{3n+2} \right\}_{n=1}^{+\infty}$

(b)  $\left\{ \frac{1-(-1)^n}{n^2} \right\}_{n=1}^{+\infty}$

(c)  $\left\{ \frac{(-1)^{n-1}}{2 \cdot 4 \cdot 6 \cdots 2n} \right\}_{n=1}^{+\infty}$

(d)  $\left\{ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \right\}_{n=1}^{+\infty}$

(e)  $\left\{ \frac{(-1)^{n+1}}{n!} \right\}_{n=1}^{+\infty}$

(f)  $\left\{ \frac{\cos nx}{x^2 + n^2} \right\}_{n=1}^{+\infty}$

(g)  $\left\{ \frac{(2x)^{n-1}}{(2n-1)^5} \right\}_{n=1}^{+\infty}$

(h)  $\left\{ \frac{(-1)^n x^{2n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right\}_{n=1}^{+\infty}$

2. Write the first five terms of each of the sequence. Determine whether the sequence converges, and if so find the limit.

(a)  $\left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty}$

(b)  $\left\{ \frac{(-1)^{n+1}}{n!} \right\}_{n=1}^{+\infty}$

(c)  $\left\{ \frac{\sqrt{n}}{n+1} \right\}_{n=1}^{+\infty}$

(d)  $\left\{ \frac{(n+1)(n+2)}{2n^2} \right\}_{n=1}^{+\infty}$

(e)  $\left\{ \frac{\pi^n}{4^n} \right\}_{n=1}^{+\infty}$

(f)  $\left\{ \frac{\ln n}{n} \right\}_{n=1}^{+\infty}$

(g)  $\left\{ \ln \left( \frac{1}{n} \right) \right\}_{n=1}^{+\infty}$

(h)  $\left\{ \left( 1 - \frac{2}{n} \right)^n \right\}_{n=1}^{+\infty}$

3. Express the sequence in the notation  $\{a_n\}_{n=1}^{+\infty}$ . Determine whether the sequence converges, and if so find its limit.

(a)  $-\frac{1}{5}, \frac{3}{8}, -\frac{5}{11}, \frac{7}{14}, -\frac{9}{17}, \dots$

(b)  $1, 0, 1, 0, 1, \dots$

(c)  $\frac{2}{3}, 0, \frac{3}{4}, 0, \frac{4}{5}, \dots$

(d)  $0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \frac{4}{5^2}, \dots$

(e)  $3, \frac{3}{2}, \frac{3}{2^2}, \frac{3}{2^3}, \frac{3}{2^4}, \dots$

(f)  $(1 - \frac{1}{2}), (\frac{1}{2} - \frac{1}{3}), (\frac{1}{3} - \frac{1}{4}), (\frac{1}{4} - \frac{1}{5}), (\frac{1}{5} - \frac{1}{6}), \dots$

(g)  $(\sqrt{2} - \sqrt{3}), (\sqrt{3} - \sqrt{4}), (\sqrt{4} - \sqrt{5}), (\sqrt{5} - \sqrt{6}), \dots$

4. Let  $\{a_n\}$  be the sequence for which  $a_1 = \sqrt{6}$  and  $a_{n+1} = \sqrt{6 + a_n}$  for  $n \geq 1$ .
- (a) Find the first 6 terms of the sequence.
- (b) It can be shown that the sequence  $\{a_n\}$  converges. Assuming this to be so, find its limit  $\ell$ .
- [Hint:  $\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} a_{n+1}$ .]

5. The *Fibonacci sequence* is defined by  $a_{n+2} = a_{n+1} + a_n$  for  $n \geq 1$ , where  $a_1 = 1$ ,  $a_2 = 1$ .
- (a) Find the first 8 terms of the sequence.
- (b) Find  $\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n}$  assuming that this limit exists.
- [Hint:  $\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \frac{a_{n+2}}{a_{n+1}}$ .]

6. Consider the sequence  $\{a_n\}_{n=1}^{+\infty}$  where

$$a_n = \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}.$$

- (a) Write out the first five terms of the sequence.
- (b) Find the limit of the sequence.
- [Hint: Sum up the terms in the formula for  $a_n$ .]
7. A sequence has its  $n$ th term given by  $a_n = \frac{3n-1}{4n+5}$ .
- (a) Write the 1st, 5th, 10th, 100th, 1000th, 10,000th and 100,000th terms of the sequence in decimal form. Make a *guess* as to the limit of this sequence as  $n \rightarrow \infty$ .
- (b) Using the definition of limit to verify that the guess in (a) is actually correct.
8. Find the least positive integer  $N$  such that  $\left| \frac{3n+2}{n-1} - 3 \right| < \epsilon$  for all  $n > N$  if
- (a)  $\epsilon = 0.01$ , (b)  $\epsilon = 0.001$ , (c)  $\epsilon = 0.0001$ .

9. If  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ , prove that  $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$ .

10. If  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ , prove that  $\lim_{n \rightarrow \infty} a_n b_n = AB$ .

11. Prove that if  $\lim_{n \rightarrow \infty} a_n$  exists, it must be unique.

12. Using the definition of limit, prove that

(a) the sequence  $\left\{ \frac{1}{n} \right\}_{n=1}^{+\infty}$  converges to 0.

(b) the sequence  $\left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty}$  converges to 1.

(c) the sequence  $\left\{ \frac{2n-1}{3n+4} \right\}_{n=1}^{+\infty}$  converges to  $\frac{2}{3}$ .

(d) the sequence  $\left\{ \frac{4-2n}{3n+2} \right\}_{n=1}^{+\infty}$  converges to  $-\frac{2}{3}$ .

13. Evaluate each of the following, using theorems of limits.

$$(a) \lim_{n \rightarrow \infty} \frac{3n^2 - 5n}{5n^2 + 2n - 6}.$$

$$(b) \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}).$$

$$(c) \lim_{n \rightarrow \infty} \left( \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1} \right).$$

$$(d) \lim_{n \rightarrow \infty} \frac{3n^2 + 4n}{2n - 1}.$$

$$(e) \lim_{n \rightarrow \infty} \left( \frac{2n-3}{3n+7} \right)^4.$$

$$(f) \lim_{n \rightarrow \infty} \frac{1 + 2 \cdot 10^n}{5 + 3 \cdot 10^n}.$$

$$(g) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n).$$

$$(h) \lim_{n \rightarrow \infty} \frac{3n^2 + 4n}{2n - 1}.$$

14. Prove that the sequence with  $n$ th term  $u_n = \frac{2n-7}{3n+2}$ ,

- (a) is monotonic increasing,
- (b) is bounded above,
- (c) is bounded below,
- (d) is bounded,
- (e) has a limit.

15. Prove that the sequence with  $n$ th term  $u_n = \frac{\sqrt{n}}{n+1}$ ,

- (a) is monotonic decreasing,
- (b) is bounded below,
- (c) is bounded above,
- (d) has a limit.

16. Determine whether the given sequence  $\{a_n\}$  is monotone by examining  $a_n - a_{n+1}$ . If so, classify it as increasing, decreasing, nonincreasing, or nondecreasing.

$$(a) \left\{ \frac{1}{n} \right\}_{n=1}^{+\infty}.$$

$$(b) \left\{ 1 - \frac{1}{n} \right\}_{n=1}^{+\infty}.$$

$$(c) \left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty}.$$

$$(d) \left\{ \frac{n}{4n-1} \right\}_{n=1}^{+\infty}.$$

$$(e) \{n - 2^n\}_{n=1}^{+\infty}.$$

$$(f) \{n - n^2\}_{n=1}^{+\infty}.$$

17. Determine whether the given sequence  $\{a_n\}$  is monotone by examining  $a_{n+1}/a_n$ . If so, classify it as increasing, decreasing, nonincreasing, or nondecreasing.

$$(a) \left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty}.$$

$$(b) \left\{ \frac{n}{2^n} \right\}_{n=1}^{+\infty}.$$

$$(c) \left\{ \frac{n}{e^n} \right\}_{n=1}^{+\infty}.$$

$$(d) \left\{ \frac{n^2}{3^n} \right\}_{n=1}^{+\infty}.$$

$$(e) \left\{ \frac{2^n}{n!} \right\}_{n=1}^{+\infty}.$$

$$(f) \left\{ \frac{e^n}{n!} \right\}_{n=1}^{+\infty}.$$

18. Use differentiation to show the sequence  $\{a_n\}$  is strictly monotone and classify it as increasing or decreasing.

(a)  $\left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty}$

(b)  $\left\{ 3 - \frac{1}{n} \right\}_{n=1}^{+\infty}$

(c)  $\left\{ \frac{1}{n + \ln n} \right\}_{n=1}^{+\infty}$

(d)  $\left\{ \frac{\ln(n+2)}{n+2} \right\}_{n=1}^{+\infty}$

(e)  $\{ne^{-2n}\}_{n=1}^{+\infty}$

(f)  $\{\tan^{-1} n\}_{n=1}^{+\infty}$

19. Find the limit (if it exists) of the sequence.

(a)  $1, -1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

(b)  $-\frac{1}{2}, 0, 0, 0, 1, 2, 3, 4, \dots$

20. Show that  $\left\{ \frac{3^n}{1+3^{2n}} \right\}_{n=1}^{+\infty}$  is a decreasing sequence.

21. Show that  $\left\{ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \right\}_{n=1}^{+\infty}$  is an increasing sequence.

## Answers for Tutorial 1

1. (a)  $1/5, 3/8, 5/11, 7/14, 9/17$  (b)  $2, 0, 2/9, 0, 2/25$  (c)  $1/2, -1/8, 1/48, -1/384, 1/3840$

(d)  $1/2, 3/4, 7/8, 15/16, 31/32$  (e)  $1, -1/2, 1/6, -1/24, 1/120$

(f)  $\cos x / (x^2 + 1), \cos 2x / (x^2 + 4), \cos 3x / (x^2 + 9), \cos 4x / (x^2 + 16),$   
 $\cos 5x / (x^2 + 25)$

(g)  $1, 2x/3^5, 4x^2/5^5, 8x^3/7^5, 16x^4/9^5$

(h)  $-x, x^3/3, -x^5/15, x^7/105, -x^9/945$

2. (a)  $1/2, 2/3, 3/4, 4/5, 5/6$ ; 1 (b)  $1, -1/2, 1/6, -1/24, 1/120$ ; 0

(c)  $1/2, \sqrt{2}/3, \sqrt{3}/4, 2/5, \sqrt{5}/6$ ; 0 (d)  $3, 3/2, 10/9, 15/16, 21/25$ ;  $1/2$

(e)  $\pi/4, \pi^2/16, \pi^3/64, \pi^4/128, \pi^5/512$ ; 0 (f)  $0, (\ln 2)/2, (\ln 3)/3, (\ln 4)/4, (\ln 5)/5$ ; 0

(g)  $0, \ln(1/2), \ln(1/3), \ln(1/4), \ln(1/5)$ ;  $-\infty$  (h)  $-1, 0, 1/27, 1/16, 243/3125$ ;  $e^{-2}$

3. (a)  $a_n = (-1)^n(2n-1)/(3n+2)$ ; diverges (b)  $a_n = (1 + (-1)^{n+1})/2$ ; diverges

(c)  $a_n = [(n+3)/(n+5)](1 + (-1)^{n+1})/2$ ; diverges (d)  $(n-1)/n^2$ ; converges; 0

(e)  $a_n = 3(1/2^{n-1})$ ; converges; 0 (f)  $a_n = 1/n - 1/(n+1)$ ; converges; 0

(g)  $a_n = \sqrt{n+1} - \sqrt{n+2}$ ; converges; 0

4. (a)  $2.4495, 2.9068, 2.9844, 2.9974, 2.9996, 2.9999, 2.9999$  (b) 3

5. (a)  $1, 1, 2, 3, 5, 8, 13, 21$  (b)  $(1 + \sqrt{5})/2$

6. (a)  $1, 3/4, 2/3, 5/8, 3/5$  (b)  $1/2$

7. (a)  $2/9, 14/25, 29/45, 299/405, 2999/4005, 29999/40005, 299999/400005$ ;  $3/4$

8. (a) 501 (b) 5001 (c) 50001 13 (a)  $3/5$  (b) 0 (c) 1 (d)  $3/2$  (e)  $16/81$  (f)  $2/3$  (g)  $1/2$   
 (h)  $\infty$

16. All are monotonic (a) dec (b) inc (c) inc (d) dec (e) dec (f) dec

17. (a) inc (b) non inc (c) dec (d) and (f) non monotonic (e) non inc

18. (a), (b), and (f) are increasing (c), (d), and (e) decreasing

19. (a) 0 (b) does not exist. The limit is  $\infty$