1 Introduction

1.1 What is a Dynamical System?

Dynamical systems occur in many branches of science and engineering, and are essentially processes which evolve in time. Examples include the motion of the stars in the heavens, the weather, variations in the stock market, some chemical reactions, population growth and decline, and the motion of a simple pendulum etc. In mathematical terms, the study of dynamical systems involves developing mathematical descriptions of such processes and their evolution in time. Often, a mathematical model of such a process takes the form of either a **differential equation** such as

$$\dot{x} = f(x(t)), \quad x(0) = x_0 \tag{1.1}$$

(where $\dot{x}$ denotes the derivative dx/dt) or a **difference equation**, e.g.

$$x_{n+1} = f(x_n), \quad x_0 \text{ specified.} \tag{1.2}$$

As we shall see, differential equations lead to **continuous time dynamical systems** in which time progresses smoothly, while difference equations give rise to **discrete time dynamical systems** in which time progresses in discrete packets. Only one-dimensional situations will be covered in this course.

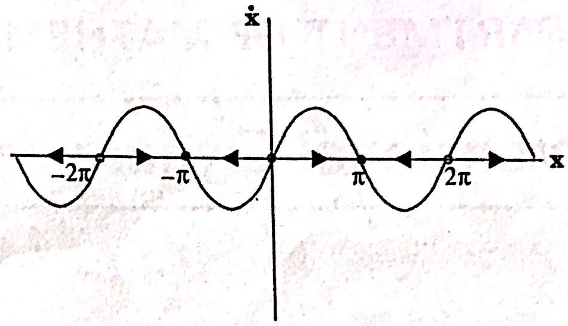
To get a flavour of what this course is about, we consider the simple one-dimensional nonlinear differential equation

$$\dot{x} = \sin x, \qquad x = x_0 \text{ at } t = 0. \tag{1.3}$$

This can be solved analytically by separation of variables to give

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|. \tag{1.4}$$

Although this result is exact, it is a bit difficult to interpret. Suppose, for example, we want to know what happens as $t \to \infty$ when $x_0 = \pi/4$, or, more generally, what is the behaviour of $x(t)$ as $t \to \infty$ for an arbitrary x_0? These questions can't easily be answered from (1.4). Now suppose that we think of t as time, x as the position of an imaginary particle moving along the real line, with $\dot{x}$ as its velocity. Then (1.3) represents a **vector field** on the line which can be plotted as $\dot{x}$ versus x, with arrows representing the direction of the velocity vector at each x (so the arrows point to the right when $\dot{x} > 0$ and to the left when

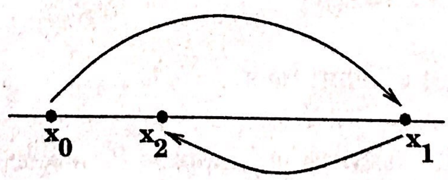


We can think of this vector field as representing a **flow** whose velocity varies according to (1.3). So the flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. When $\dot{x} = 0$, there is no flow: such points are called **fixed points**. Solid black dots represent **stable** fixed points (often called **attractors** or **sinks** because the flow is towards them) and open circles represent **unstable** fixed points (also known as **repellers** or **sources**). Using this picture, consider a particle starting at $x_0 = \pi/4$. It will move to the right, eventually approaching the stable fixed point π from the left. We can use this approach for any choice of x_0. Although the picture can't tell us some *quantitative* things (like the maximum speed), it can be used to get lots of *qualitative* information about the flow's general behaviour.

Similar ideas can be used to study dynamical systems in which time is discrete. For example, the rule

$$x_{n+1} = \cos x_n$$

is an example of a one-dimensional **map** or **difference equation iteration** (start from some number x_0 on your calculator and repeatedly press the cosine button). The sequence of iterates $x_0, x_1, x_2, \ldots$ is called the **orbit** starting from x_0. As well as being useful tools for analysing differential equations, maps are interesting in their own right as they are capable of much wilder behaviour than differential equations as points 'hop' along their orbits instead of flowing continuously.



Interesting questions can be asked re for example the behaviour of the iterates as $t \to \infty$.

On this course, we will look at the mathematical background behind such analyses of dynamical systems.

1.2 Mathematical Definition

From a mathematical viewpoint, a dynamical system consists of two parts:

the left when

- a **state vector** which describes the current (initial) state of the system (in one dimension, this is simply a scalar),

- a **function** which maps the state at one instant of time to the state at a later time.

The following definition expresses this more precisely.

Definition 1.1 *Let X be any space and let $T \subseteq \mathbb{R}$. Any function $\psi : X \times T \to X$ that has the two properties*

(i) $\psi(x, 0) = x$ (initial state)

(ii) $\psi(\psi(x,t), s) = \psi(x, t+s)$ *(the* ***semigroup property****)*

is called a **dynamical system** *on X.*

Remarks

1. Space X might be any *normed vector space* such as $\mathbb{R}^n$, or any *metric space* (see Chapter 2). In this course, we will concentrate on one–dimensional dynamical systems which means that the space X is usually $\mathbb{R}$.

2. We can regard $\psi(x,t)$ as *the state at time t of the system that initially was at state x.* The semigroup property then has the following interpretation: let the system evolve from its initial state x to state $\psi(x,t)$ at time t, and then allow it to evolve from this state for a further time s. The system will then arrive at precisely the state $\psi(x, t+s)$ that it would have reached through a single–stage evolution of $t+s$ from state x.

3. There are two main possibilities for T. Firstly, if $T = \mathbb{R}^+$ (the set of non-negative real numbers), we are dealing with continuous time so we have a **continuous dynamical system** (CDS) or **flow**. Similarly, if $T = \mathbb{N}$ (the set of non-negative integers), we are dealing with discrete time or a **discrete dynamical system** (DDS) (see below). Note that in some examples, we may also have $T = \mathbb{R}$ (CDS) or $T = \mathbb{Z}$ (DDS).

For a DDS, we suppose that the evolution through time of a particular system occurs in discrete steps, e.g. in steps of size Δt. If we write $\psi(x, n)$ to denote the value at time $t = n\Delta t$ of the system that took the value x at $t = 0$, then, for a one-dimensional DDS, ψ is defined on $\mathbb{R} \times \mathbb{N}$. Any such function ψ satisfying

$n = 0 \Rightarrow f^0(x) = x$ – start initial condition

(i) $\psi(x, 0) = x$, $\forall x \in \mathbb{R}$

(ii) $\psi(\psi(x,n),m) = \psi(x,n+m) \quad \forall x \in \mathbb{R}, \forall n,m \in \mathbb{N}$

defines a discrete dynamical system.

CDS Example

As a simple illustration of how a CDS arises from a differential equation, consider the initial value problem

$$\dot{y}(t) = ay(t)\,, \quad y(0) = x\,. \tag{1.5}$$

It is straightforward to show that the solution to (1.5) is $y(t) = xe^{at}$. Now let $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(x,t) = xe^{at}\,, \quad x,t \in \mathbb{R}\,,$$

that is, $\psi(x,t)$ denotes the value at time t of the solution of the IVP (1.5). Clearly

(i) $\psi(x,0) = x$

(ii) $\psi(\psi(x,t),s) = \psi(\underset{x}{\underline{xe^{at}}}, \underset{t}{\underline{s}}) = \underset{x}{\underbrace{xe^{at}}} e^{\overset{t}{as}} = xe^{a(t+s)} = \psi(x,t+s)$

so $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a one–dimensional CDS (by Definition 1.1).

DDS Example

As an example of how a one-dimensional DDS might be generated, consider the function (or map) $f : \mathbb{R} \to \mathbb{R}$ which satisfies the first–order difference equation (or iteration)

$$x_{n+1} = f(x_n), \quad x_0 \text{ specified}\,.$$

For $n \in \mathbb{N}$, we define the nth **iterate** or n–**fold composition** of f to be

$$f^n = f \circ f \circ f \ldots \circ f \quad (n \text{ terms}).$$

Note that f^n does not mean "f to the power n" here, but n applications of f:

$$f^2(x) = f(f(x)), \qquad f^3(x) = f(f^2(x)) = f(f(f(x))), \qquad \text{etc.}$$

If we also define f^0 by $f^0(x) = x \quad \forall x \in \mathbb{R}$, it then follows that

(i) $f^0(x) = x \quad \forall x \in \mathbb{R}$

(ii) $f^n(f^m(x)) = f^{n+m}(x) \quad \forall x \in \mathbb{R}, \forall n,m \in \mathbb{N}.$

Writing

$$\psi(x,n) = f^n(x) \; \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$

we see that $\psi(x,n)$ satisfies the properties of a discrete dynamical system.

We mention here another important way of generating a DDS. Suppose that $\phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a CDS. If we restrict the times to discrete values, say $t = 0, \Delta t, 2\Delta t, \ldots$, and define $\psi(x, n) \equiv \phi(x, n\Delta t)$, then we obtain a DDS $\psi : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$. This method of "strobing " a CDS is a very useful technique for analysing its behaviour, and is essentially due to Henri Poincaré.

1.3 Main Objectives

When studying a given CDS or DDS $\psi : X \times T \to X$, there are usually several questions that we would like to answer:

1. Given an initial value x, can we determine the asymptotic (long–term) behaviour of $\psi(x, t)$ as $t \to \infty$?
2. Can we identify particular initial values which give rise to the same asymptotic behaviour?
3. Can we say anything about the stability of the system? That is, if x is "close to" y in X, is it true that $\psi(x, t)$ is "close to" $\psi(y, t) \, \forall t \in T$?

In many situations, a dynamical system may also depend on a parameter, that is, the system takes the form $\psi_\mu : X \times T \to X$ where $\mu \in \mathbb{R}$ represents the parameter. In such cases, the following questions would also be of interest:

4. Can we determine what happens to the behaviour of the dynamical system as the parameter varies?
5. Can we identify the values of the parameter at which changes in the behaviour of the system occur (**bifurcation values**)?

In some special cases, it is possible to find an explicit formula for the dynamical system (e.g. in the example in §1.2 where $\psi_a(x, t) = xe^{at}$ with parameter a). This can then be used to answer questions 1 – 5 above. Unfortunately, in most cases no such formula can be found and analysing the dynamical system becomes more complicated.

1.4 Some Motivating Examples

Example 1.1 ***Dynamical systems in ecology.*** *Suppose we are interested in the long-term behaviour of the population of a particular species (or collection of species). A mathematical model can be developed that contains certain observed or experimentally determined* parameters such as the number of predators, severity of climate, availability of food e... model may take the form of a differential equation or a difference equation depending ... whether the population is assumed to change continuously or discretely. We can attempt to use the model to answer questions such as:

1. Does the population $\to 0$ as $t \to \infty$ (**extinction**)?
2. Does the population become arbitrarily large as $t \to \infty$ (**eventual overcrowding**)?
3. Does the population fluctuate periodically or even randomly?

1. (a) **Single species – continuous models.**

   (i) **Linear case.** Let $P(t)$ represent the size of the population at time $t \geq 0$, with initial population $P(0) = P_0$. A simple model is given by the linear differential equation

$$\frac{dP}{dt} = \mu P, \quad t > 0$$

   where μ is the (constant) growth rate. Solving this equation we obtain

$$P(t) = P_0\, e^{\mu t}, \quad t \geq 0$$

   from which we can deduce that:

$$\begin{array}{lll} \mu > 0 \Longrightarrow P(t) \to \infty & \text{as } t \to \infty & \text{(overcrowding)} \\ \mu < 0 \Longrightarrow P(t) \to 0 & \text{as } t \to \infty & \text{(extinction)} \\ \mu = 0 \Longrightarrow P(t) = P_0 & \forall t \geq 0. & \end{array}$$

   Note that in the first two cases, the long-term behaviour of $P(t)$ is independent of P_0. Also, for $\mu < 0$, extinction will occur in **finite** time since fractional values of $P(t)$ are not allowed.

   (ii) **Nonlinear case.** A more realistic model is given by

$$\frac{dP}{dt} = G(P)P, \quad t > 0; \quad P(0) = P_0,$$

   with a variable growth rate G depending on the population P. For example, the **equation of limited growth** is

$$\frac{dP}{dt} = P(\mu - \lambda P), \quad P(0) = P_0 \quad (\mu, \lambda \text{ positive constants}).$$

   If we write $x = \lambda P/\mu$ then this reduces to

$$\frac{dx}{dt} = \mu x(1 - x), \quad x(0) = x_0$$

$P = \frac{\mu}{\lambda} x$

$x = \frac{\lambda}{\mu} P$

$\frac{dx}{dt} = \frac{\lambda}{\mu} \frac{dP}{dt}.$

$\therefore \frac{\mu}{\lambda} \frac{dx}{dt} = \frac{\mu x}{\lambda} \left(\mu - \cancel{\lambda} \frac{\mu x}{\cancel{\lambda}}\right)$

$\frac{dx}{dt} = \mu x(1-x)$ *

which has solution $x(t) = x_0 e^{\mu t}/(1 - x_0 + x_0 e^{\mu t})$.

Thus,

$$x_0 > 1 \implies x(t) \to 1 \text{ from above as } t \to \infty$$
$$0 < x_0 < 1 \implies x(t) \to 1 \text{ from below as } t \to \infty$$
$$x_0 = 1 \implies x(t) = 1 \ \forall t \geq 0.$$

$$x(t) = \frac{x_0 e^{\mu t}}{1 - x_0 + x_0 e^{\mu t}} = \frac{1}{1 + \frac{1 - x_0}{x_0 e^{\mu t}}}.$$

(b) *Single species – discrete models.*

(i) **Linear case.** *Let P_n represent the population after n generations with the initial population given by P_0. A simple discrete model is the linear difference equation*

$$P_{n+1} = \mu P_n \quad (\mu > 0)$$

which has solution $P_n = P_0 \mu^n$.

Thus:

$$\mu > 1 \implies P_n \to \infty \text{ as } n \to \infty,$$
$$0 < \mu < 1 \implies P_n \to 0 \text{ as } n \to \infty,$$
$$\mu = 1 \implies P_n = P_0 \ \forall n.$$

(ii) **Nonlinear case.** *A slightly more realistic discrete model is the quadratic model*

$$P_{n+1} = P_n(\mu - \lambda P_n), \quad P_0 \text{ specified.}$$

By setting $x_n = \lambda P_n/\mu$ we reduce this to

logistic map

$$\boxed{x_{n+1} = \mu x_n(1 - x_n) = f_\mu(x_n)} \tag{1.6}$$

where x_0 is specified. Equation (1.6) is known as the **logistic equation** *and $f_\mu(x) = \mu x(1-x)$ is called the* **logistic function**.