Dynamical Systems

Part A: Continuous Dynamical Systems

Part B: Discrete Dynamical Systems

What is a Dynamical System?

Dynamical systems occur in many branches of science and engineering, and are essentially processes which evolve in time. Examples include the motion of the stars, the weather, variations in the stock market, some chemical reactions, population growth and decline, and the motion of a simple pendulum etc. In mathematical terms, the study of dynamical systems involves developing mathematical descriptions of such processes and their evolution in time. Often, a mathematical model of such process takes the form of either differential equation such as

$$\frac{dx}{dt} = f\big(x(t)\big), \quad x(0) = x_0$$

or a difference equation, e.g

$$x_{n+1} = f(x_n), \qquad x_0 \text{ specified.}$$

Chapter 1: Phase Portraits with Emphasis on Fixed points.

1.1 Stability of Fixed Points

Give a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

has a continuous partial derivatives of the components of $\mathbf{F}$, so the solutions exist and unique. Let $\phi(t; \mathbf{x}_0)$ be the *flow* (*trajectory*); that is

$$\frac{d}{dt}\phi(t; \mathbf{x}_0) = \mathbf{F}(\phi(t; \mathbf{x}_0)) \text{ and}$$
$$\phi(0; \mathbf{x}_0) = \mathbf{x}_0$$

Definition: A point $\mathbf{x}^*$ is called a *fixed point*, provided that $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$. The solution starting at a fixed point has zero velocity, so it stays there and $\phi(t; \mathbf{x}^*) = \mathbf{x}^*$ for all *t*.

Definition: For each $\mathbf{x_0} \in \mathbb{R}^n$, we define the ***orbit*** $\gamma(\mathbf{x_0})$ through $\mathbf{x_0}$ by

$$\gamma(\mathbf{x_0}) = \{\phi(t; \mathbf{x}_0) | \mathrm{t} \in \mathbb{R}\}.$$

Similarly,

$$\gamma_+(\mathbf{x_0}) = \{\phi(t; \mathbf{x}_0) | \mathrm{t} \geq 0\}, \quad \gamma_-(\mathbf{x_0}) = \{\phi(t; \mathbf{x}_0) | \mathrm{t} \leq 0\}$$

Are the *positive* and *negative* semi-orbits through $\mathbf{x_0}$.

Definition: A point **q** is an ω-limit point of the trajectory of $\mathbf{x}_0$, provided that $\phi(t; \mathbf{x}_0)$ keeps coming near **q** as *t* goes to infinity. Certainly, if $\|\phi(t; \mathbf{x}_0) - \mathbf{x}^*\|$ goes to zero as *t* goes to infinity, then $\mathbf{x}^*$ is the only ω-limit point of $\mathbf{x}_0$. The set of all ω-limit points is denoted by $\omega(\mathbf{x}_0)$ and is called ω-limit set of $\mathbf{x}_0$.

Definition: Similarly, a point **q** is an α-limit point of the trajectory of $\mathbf{x}_0$, provided that $\phi(t; \mathbf{x}_0)$ keeps coming near **q** as *t* goes to minus infinity. In particular, if $\|\phi(t; \mathbf{x}_0) - \mathbf{x}^*\|$ goes to zero as *t* goes to minus infinity, then $\mathbf{x}^*$ is the only α-limit point of $\mathbf{x}_0$. The set of all α-limit points is denoted by $\alpha(\mathbf{x}_0)$ and is called α-limit set of $\mathbf{x}_0$.

Definition: For a fixed point $\mathbf{x}^*$, the *stable manifold* $W^s(\mathbf{x}^*)$ is the set of all points which tend to the fixed point as *t* goes to plus infinity:

$$W^s(\mathbf{x}^*) = \{\mathbf{p}_0 : \phi(t; \mathbf{p}_0) \text{tends to } \mathbf{x}^* \text{ as } t \to \infty \ \} = \{\mathbf{p}_0 : \omega(\mathbf{p}_0) = \mathbf{x}^*\}$$

If the stable manifold is an open set, then $W^s(\mathbf{x}^*)$ is called the *basin of attraction* of $\mathbf{x}^*$.

Definition: A fixed point $\mathbf{x}^*$ is said to be *Lyapunov stable* or *L-stable*, provided that any solution $\phi(t; \mathbf{x}_0)$ stays near $\mathbf{x}^*$ for all $t \geq 0$, if the initial condition $\mathbf{x}_0$ stays near $\mathbf{x}^*$. More precisely, a fixed point $\mathbf{x}^*$ is said to be *Lyapunov stable*, provided that for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$, then $\|\phi(t; \mathbf{x}_0) - \mathbf{x}^*\| < \epsilon$ for all $t \geq 0$.

Definition: A fixed point $\mathbf{x}^*$ is called *unstable*, provided that it is not *L-stable*(i.e there exists and $\epsilon_1 > 0$ such that for any $\delta > 0$ there is some point $\mathbf{x}_\delta$ with $\|\mathbf{x}_\delta - \mathbf{x}^*\| < \delta$ and a time $t_1 > 0$ depending on $\mathbf{x}_\delta$ with $\|\phi(t_1; \mathbf{x}_\delta) - \mathbf{x}^*\| > \epsilon_1$). Thus, trajectories that start as near as $\mathbf{x}^*$ as we would like to specify move at least a distance ϵ_1 away from $\mathbf{x}^*$.

Definition: A fixed point $\mathbf{x}^*$ is called *weakly asymptotically stable*, provided that there exists a $\delta_1 > 0$ such that $\omega(\mathbf{x}_0) = \{\mathbf{x}^*\}$ for all $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta_1$. (i.e $\|\phi(t; \mathbf{x}_0) - \mathbf{x}^*\|$ goes to zero as *t* goes to infinity for all $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta_1$). Thus, a fixed point is called weakly asymptotically stable, provided that the stable manifold contains all points in a neighborhood of the fixed point. (i.e all points sufficiently close)

A fixed point $\mathbf{x}^*$ is called *asymptotically stable (attracting)*, provided that it is both L-stable and weakly asymptotically stable. An asymptotically stable fixed point is also called a fixed point *sink.*

Definition: A fixed point is called *repelling* or a fixed point *source*, provided that it is asymptotically stable backward in time
(i.e (i)for any closeness $\epsilon > 0$, there is a $\delta > 0$ such that if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$, then $\|\phi(t; \mathbf{x}_0) - \mathbf{x}^*\| < \epsilon$ for all $t \leq 0$.
(ii) there exists $\delta_1 > 0$ such that $\alpha(\mathbf{x}_0) = \{\mathbf{x}^*\}$ for all $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta_1$).

Definition: A fixed point is called *hyperbolic*, provided that none of the eigenvalues at the fixed point have zero real part.

For linear system, the criteria for stability is in the theorem below
Theorem: Consider the linear differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

(a) If all of the eigenvalues λ of $\mathbf{A}$ have negative real parts, then the origin is asymptotically stable
(b) If one of the eigenvalues has a positive real part, then the origin is unstable.
(c) In two dimensions, if the eigenvalues are purely imaginary $\pm\beta i$, then the origin is L-stable but not asymptotically stable.
(d) In two dimensions, if one eigenvalue is 0 and the other is negative, then the origin is L-stable but not asymptotically stable.

1.2 One dimensional Differential Equations

Theorem: Consider a differential equation $\dot{x} = f(x)$ on $\mathbb{R}$, for which $f(x)$ has a continuous derivative. Assume that $x(t) = \phi(t{:}\, x_0)$ is the solution, with initial condition x_0. Assume that the maximum interval containing 0 for which it can be defined is (t^-, t^+).

(a) Further assume that the solution $\phi(t{:}\, x_0)$ is bounded for $0 \leq t \leq t^+$, (i.e., there is a constant $C > 0$ such that $|\phi(t{:}\, x_0)| \leq C$ for $0 \leq t \leq t^+$) . Then $\phi(t{:}\, x_0)$ must converge either to a fixed point or to a point where $f(x)$ is undefined as t converges to t^+.

(b) Similarly, if the solution $\phi(t{:}\, x_0)$ is bounded for $t^- < t \leq 0$, then $\phi(t{:}\, x_0)$ must converge to either to a fixed point or to a point where $f(x)$ is undefined as t converges to t^-.

(c) Assume that $f(x)$ is defined for all x in $\mathbb{R}$.

(i) If $f(x_0) > 0$, assume that there is a fixed point $x^* > x_0$, and in fact let x^* be the smallest fixed point larger than x_0.

(ii) If $f(x_0) < 0$, assume that there is a fixed point $x^* < x_0$, and in fact let x^* be the largest fixed point less than x_0.Then, $t^+ = \infty$ and $\phi(t{:}\, x_0)$ converges to x^* as t goes to infinity.

Lemma: Assume that, for all $t \geq 0$, $g(t)$ is defined, $g(t)$ is bounded, $g'(t) > 0$, and $g'(t)$ is uniformly continuous. Then $g'(t)$ approaches 0 and $g(t)$ approaches a limiting value as t goes to infinity.

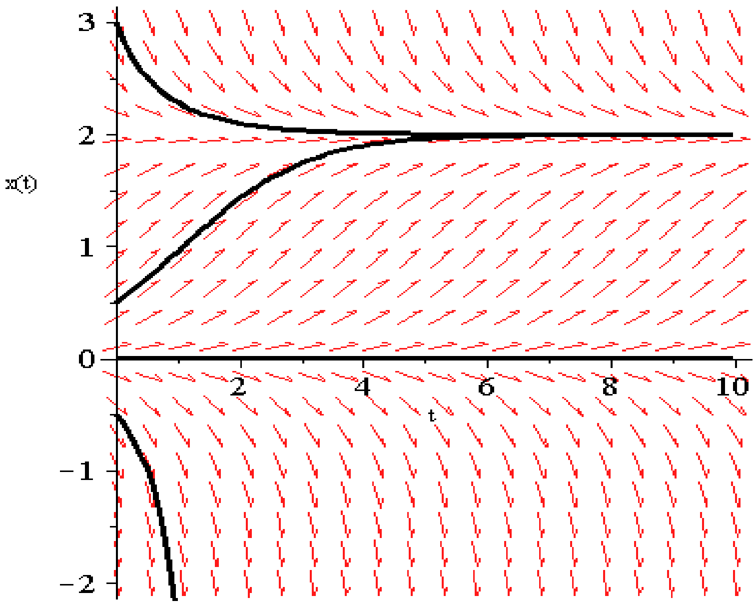
Theorem: Consider a fixed point x^* for the differential equation $\dot{x} = f(x)$ where f and f' are continuous.

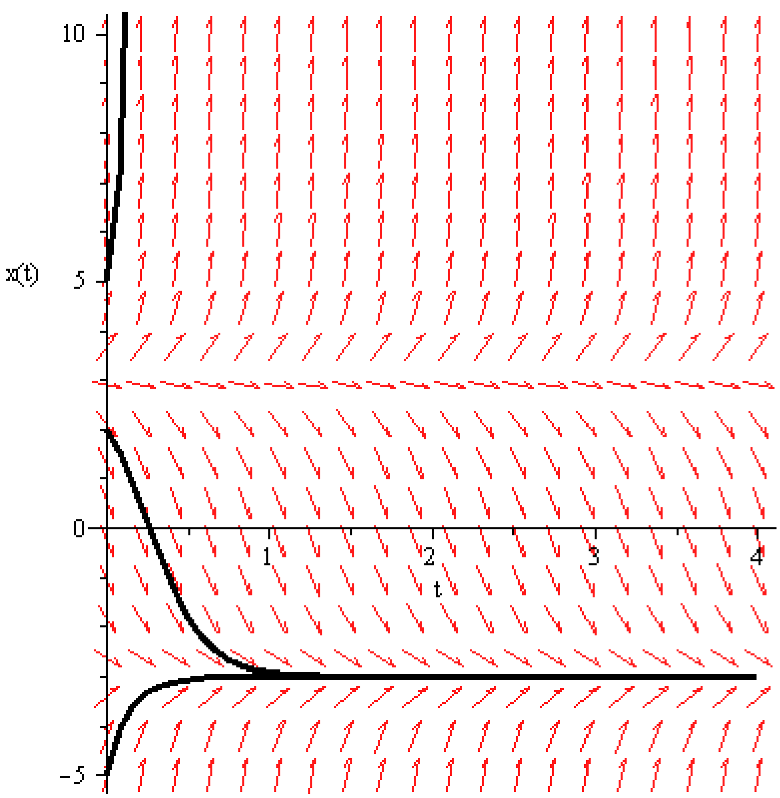
(a) If $f'(x^*) < 0$, then x^* is an attracting fixed point.
(b) If $f'(x^*) > 0$, then x^* is a repelling fixed point.
(c) If $f'(x^*) = 0$, then the derivative does not determine the stability type.

Examples: For each of the differential equation below

(a) find the stability type of each fixed point.
(b) Sketch the phase portrait on the line.
(c) Sketch the graph of $x(t) = \phi(t; x_0)$ in the (t, x)-plane for several representative initial conditions x_0.

1. $\dot{x} = rx\left(1 - \frac{x}{K}\right)$
2. $\dot{x} = x^2 - 9$
3. $\dot{x} = -x$
4. $\dot{x} = x^2$


3
2
x(t)
1
0
2
4
6
8
10
t
−1
−2


10
x(t)
5
0
1
2
3
4
t
-5

1.2 Two dimensions

Consider linear two-dimensional autonomous system of the form

$$\dot{x} = a_{11}x + a_{12}y$$
$$\dot{y} = a_{21}x + a_{22}y$$

where a_{ij} are constants. The system can be written in matrix form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

where $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Let's say, we assume the solution is

$$\boldsymbol{x} = \boldsymbol{v}e^{\lambda t}$$

where $\boldsymbol{v}$ is a matrix independent of t.

$$\dot{\boldsymbol{x}} = \lambda\boldsymbol{v}e^{\lambda t} = A\boldsymbol{v}e^{\lambda t}$$

$$\lambda\boldsymbol{v} = A\boldsymbol{v}$$

$$A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$$

$$(A - \lambda\mathbf{I})\boldsymbol{v} = 0$$

$$\det(A - \lambda\mathbf{I}) = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

The characteristic polynomial is

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

Definition: A critical point $\mathbf{x}^*$ of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2$$

at which the Jacobian matrix has no zero eigenvalues is called *nondegenerate critical point*; otherwise, it is called a degenerate critical point.

Theorem: Let A be a 2×2 matrix with determinant Δ and trace τ.

(a) If $\Delta < 0$, then the linear system is a saddle, and therefore unstable.

(b) If $\Delta > 0$ and $\tau > 0$, then the linear system is unstable.

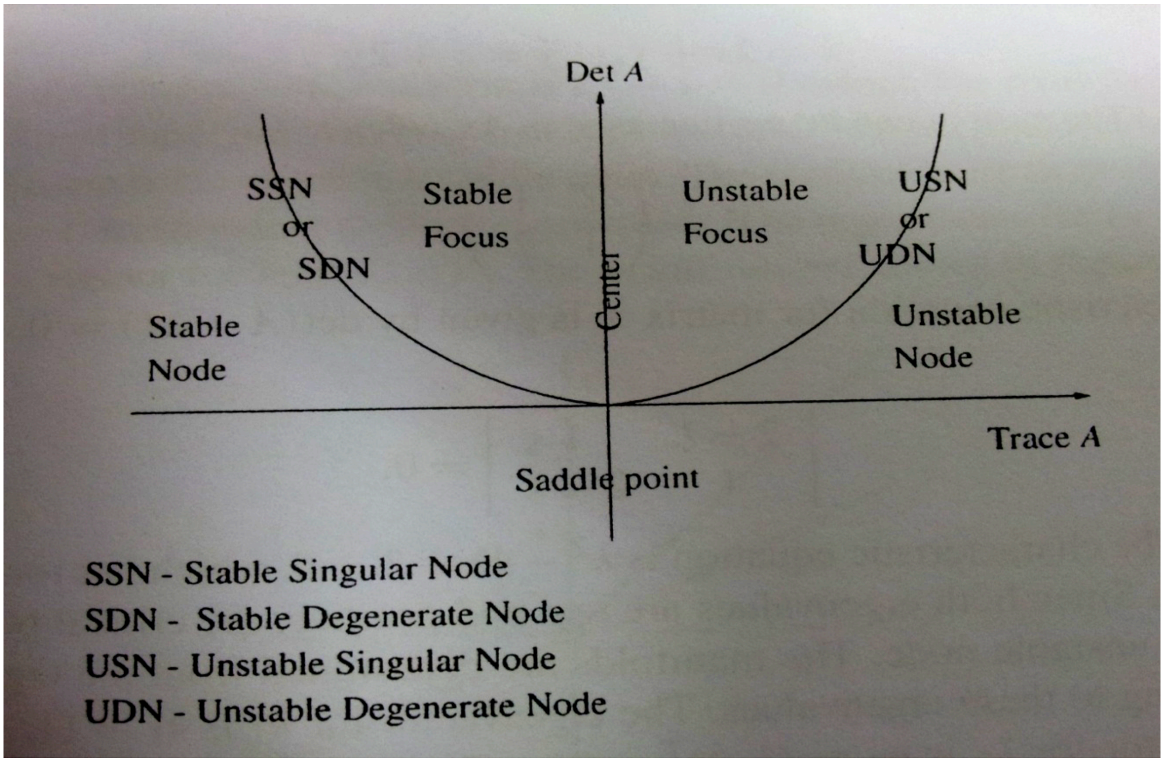
- (i) If $\tau^2 - 4\Delta > 0$, then it is an unstable node.
- (ii) If $\tau^2 - 4\Delta = 0$, then it is a degenerate unstable node.
- (iii) If $\tau^2 - 4\Delta < 0$, then it is an unstable focus.

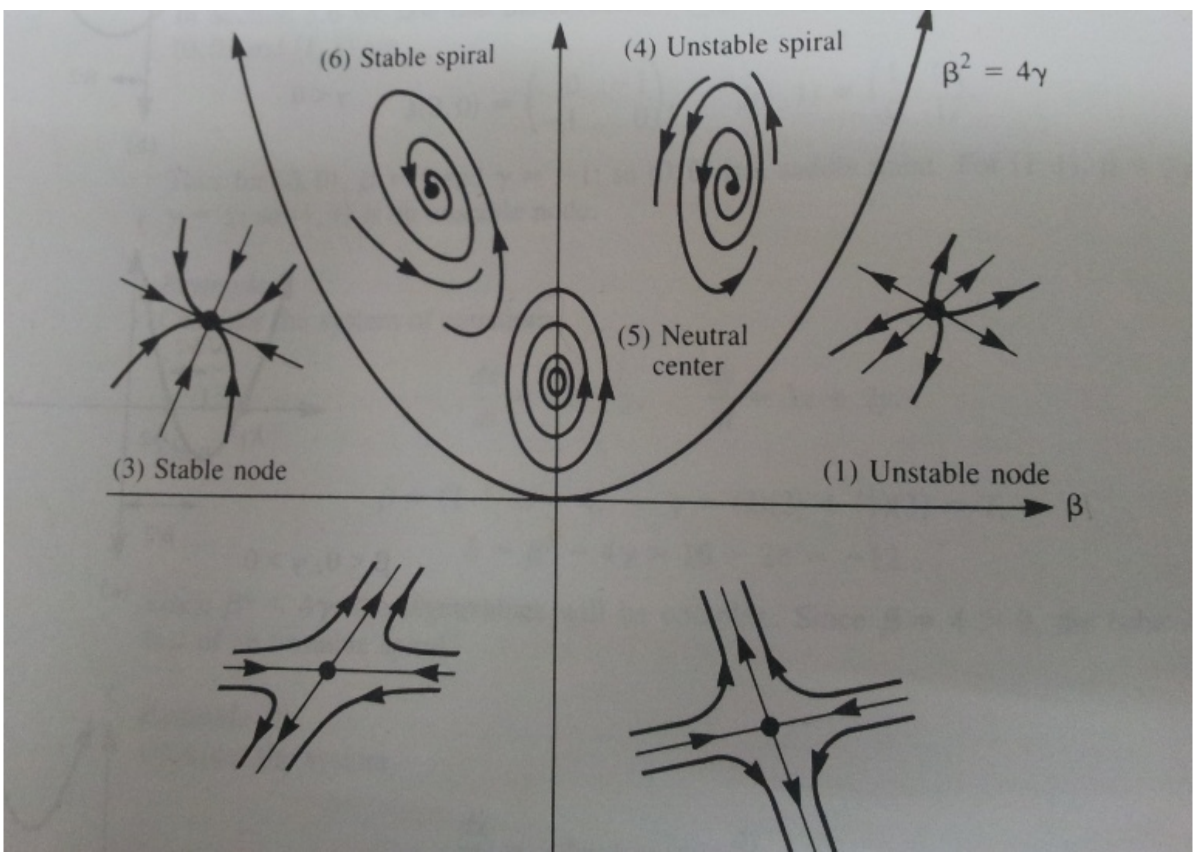
(c) If $\Delta > 0$ and $\tau < 0$, then the linear system is asymptotically stable.

- (i) If $\tau^2 - 4\Delta > 0$, then it is a stable node.
- (ii) If $\tau^2 - 4\Delta = 0$, then it is a degenerate stable node.
- (iii) If $\tau^2 - 4\Delta < 0$, then it is a stable focus.

(d) If $\Delta = 0$, then one or more of the eigenvalues is 0.

- (i) If $\tau > 0$, then the second eigenvalue is positive.
- (ii) If $\tau = 0$, then both eigenvalues are zero.
- (iii) If $\tau < 0$, then the second eigenvalue is negative.


Det A
SSN
or
SDN
Stable
Focus
Unstable
Focus
USN
or
UDN
Center
Stable
Node
Unstable
Node
Trace A
Saddle point
SSN - Stable Singular Node
SDN - Stable Degenerate Node
USN - Unstable Singular Node
UDN - Unstable Degenerate Node


(6) Stable spiral
(4) Unstable spiral
$\beta^2 = 4\gamma$
(5) Neutral
center
(3) Stable node
(1) Unstable node
β

Definition: The curves where $\dot{x} = 0$ and $\dot{y} = 0$ are called nullclines or isoclines.

Definition: The *phase portrait* is a two-dimensional figure showing how the qualitative behavior of the system is determined as x and y vary with t.

Definition: The *direction field* or *vector field* gives the gradients $\frac{dy}{dx}$ and direction vectors of the trajectories in the phase plane.

Constructing Phase plane diagram

The method of plotting phase portraits for nonlinear planar system having hyperbolic critical point may be broken into three distinct steps:

1. Locate all of the critical points
2. Linearize and classify each critical point according to Hartman's theorem
3. Determine the isoclines and use $\frac{dy}{dx}$ to obtain slopes of the trajectories.

Examples:

5. Consider the system of linear differential equations

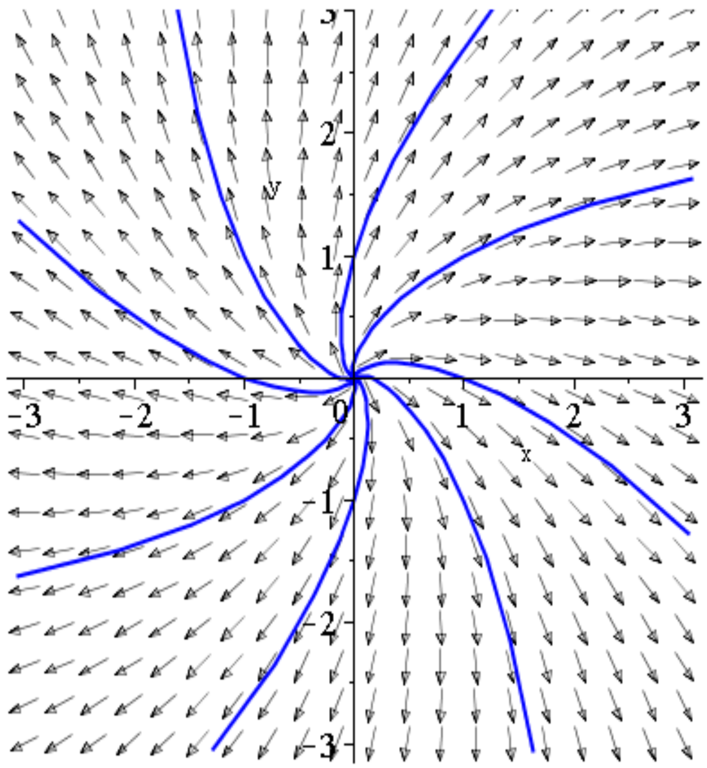
$$\dot{x} = 3x + y$$
$$\dot{y} = -x + 3y$$

   Sketch the phase portrait of the system.

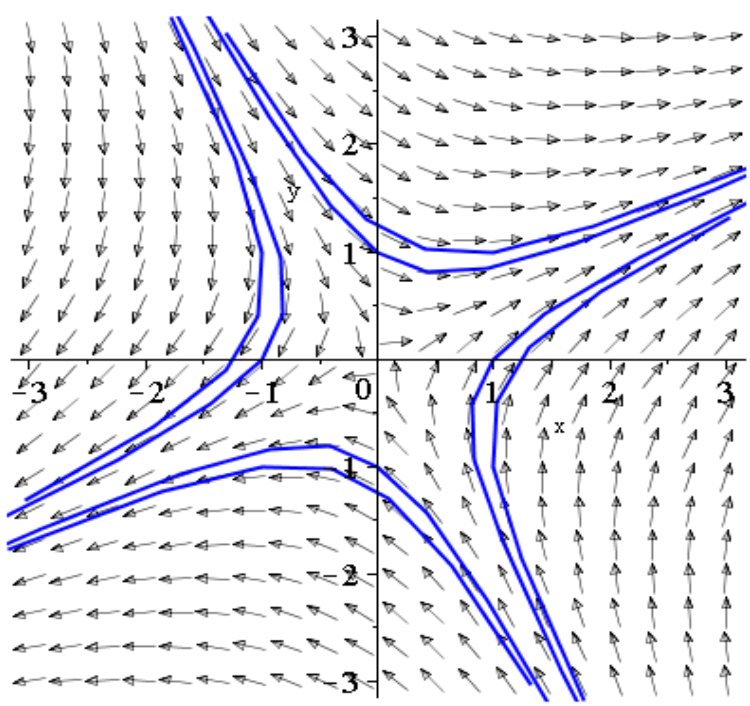
6. Consider the system of linear differential equations

$$\dot{x} = 3x + 4y$$
$$\dot{y} = 4x - 3y$$

   Sketch the phase portrait of the system.



3
2
y
1
-3
-2
-1
0
1
2
3
x
-1
-2
-3



3
2
y
1
-3
-2
-1
0
1
2
3
x
-1
-2
-3

Linearization and Hartman's Theorem

Suppose the nonlinear autonomous system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

Have a critical point at (u, v). Take a linear transformation which moves the critical point to the origin. Let $X = x - u$ and $Y = y - v$. The system becomes

$$\dot{X} = f(X+u, Y+v) = f(u, v) + X\left.\frac{\partial f}{\partial x}\right|_{x=u, y=v} + Y\left.\frac{\partial f}{\partial y}\right|_{x=u, y=v} + R(X, Y)$$

$$\dot{Y} = g(X+u, Y+v) = g(u, v) + X\left.\frac{\partial g}{\partial x}\right|_{x=u, y=v} + Y\left.\frac{\partial g}{\partial y}\right|_{x=u, y=v} + S(X, Y)$$

after a Taylor series expansion, where R and S are nonlinear terms. The linearized system at the critical point (u, v) is then of the form

$$\dot{X} == X\left.\frac{\partial f}{\partial x}\right|_{x=u, y=v} + Y\left.\frac{\partial f}{\partial y}\right|_{x=u, y=v}$$

$$\dot{Y} = X\left.\frac{\partial g}{\partial x}\right|_{x=u, y=v} + Y\left.\frac{\partial g}{\partial y}\right|_{x=u, y=v}$$

which can be written as $\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$.

$J(u, v) = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{x=u, y=v}$ is known as the Jacobian matrix.

Hartman's Theorem: Suppose that (u, v) is a hyperbolic critical point of the system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y),$$

then there is a neighborhood of this critical point on which the phase portrait for the nonlinear system resembles that of the linearized system. In other words, there is a curvilinear continuous change of coordinate taking one phase portrait to the other , and in a small region around the critical point, the portrait are qualitatively equivalent.

Examples:

7. Consider the system of differential equations given

$$\dot{x} = x^2 - y$$
$$\dot{y} = x - y$$

   (a) Determine the fixed points
   (b) Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.
   (c) Using the information from part(a) and (b), sketch by hand a rough phase portrait.

8. Consider the system of differential equations

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = xy - 1$$

   (a) Determine the fixed points
   (b) Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.
   (c) Using the information from part(a) and (b), sketch by hand a rough phase portrait.

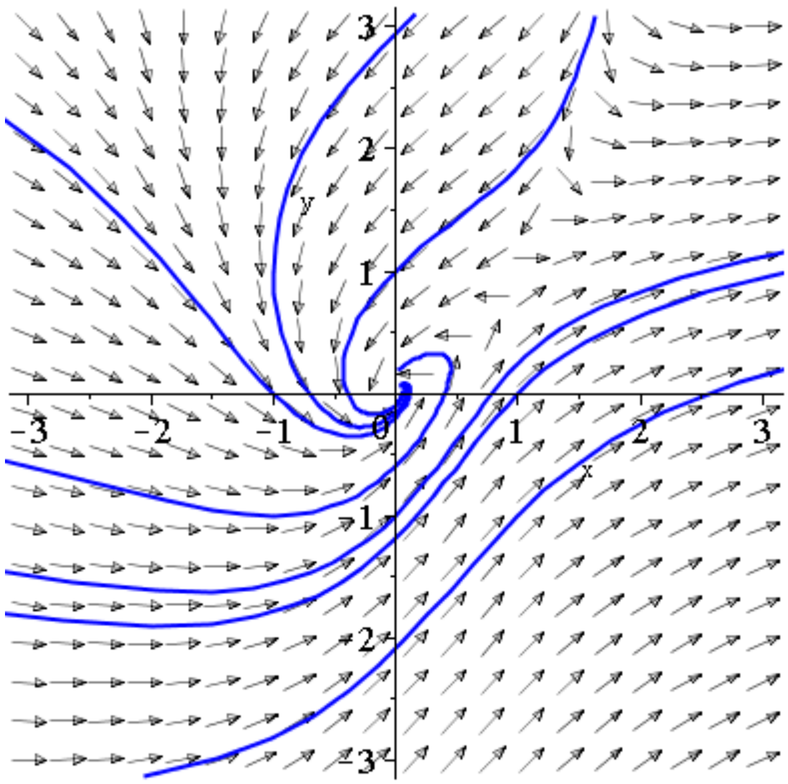
9. Consider the system of differential equations

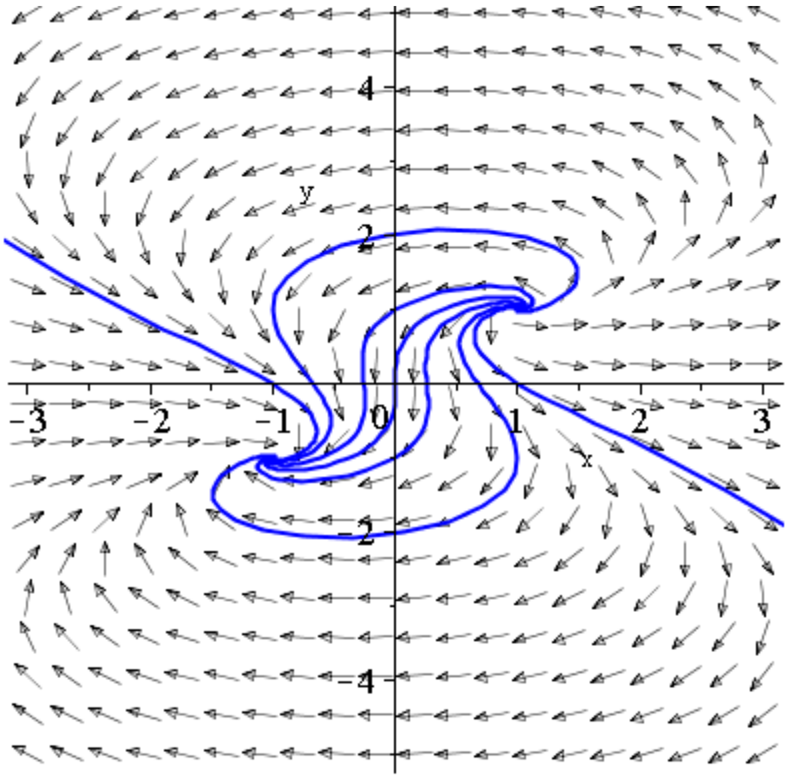
$$\dot{x} = -x + y$$
$$\dot{y} = xy - 1$$

(a)Determine the fixed points

(b)Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.

(c)Using the information from part(a) and (b), sketch by hand a rough phase portrait.





y
x
4
2
−2
−4
−3
−2
−1
0
1
2
3

1.3 Hamiltonian Systems in the Plane

Definition: A system of differential equations on $\mathbb{R}^2$ is said to be *Hamiltonian* with one degree of freedom if it can be expressed in the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

where $H(x, y)$ is a twice-continuously differentiable function. The system is said to be conservative and there is no dissipation. In applications, the Hamiltonian is defined by

$$H(x, y) = K(x, y) + V(x, y)$$

where $K(x, y)$ is the kinetic energy and $V(x, y)$ is the potential energy. The trajectories lie on the contour defined by

$$H(x, y) = C.$$

Theorem: (Conservation of Energy) The total energy $H(x, y)$ is a first integral and a constant of the motion.

Proof: The total derivative along a trajectory is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = 0$$

Therefore, $H(x, y)$ is constant along the solution curve of the Hamiltonian system, and the trajectories lie on the contour defined by $H(x, y) = C$.

Definition: A critical point $\mathbf{x}^*$ of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2$$

at which the Jacobian matrix has no zero eigenvalues is called *nondegenerate critical point*; otherwise, it is called a degenerate critical point.

Theorem: Any *nondegenerate* critical point of an analytic Hamiltonian system is either a saddle or a centre.

Examples:

7. (Double-Well Potential/Duffing equation) :

(a) Find the Hamiltonian for the given system, and sketch the phase portrait.

$$\dot{x} = y$$

$$\dot{y} = x - x^3$$

(b)

$$\dot{x} = y$$

$$\dot{y} = -x - x^2$$

(c)

$$\dot{x} = y + x^2 - y^2$$

$$\dot{y} = -x - 2xy$$

8. The motion of a pendulum in the plane is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

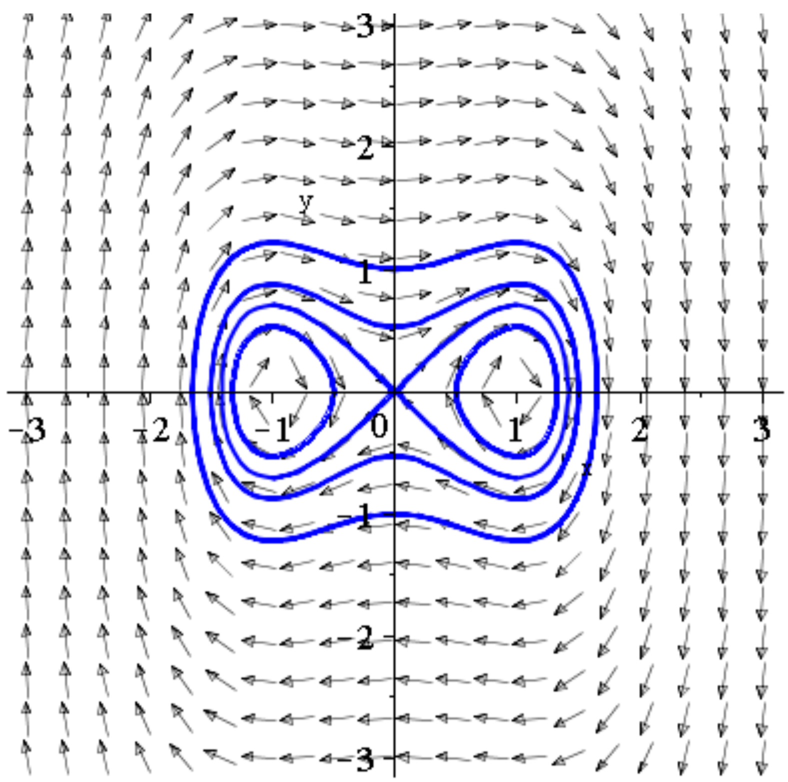
where θ is the angular displacement from the vertical, l is the length of the arm of the pendulum, which swings in the plane, and g is the acceleration due to gravity. The model does not take into account the resistive forces, so once the pendulum is set into motion, it will swing periodically forever, thus obeying the conservation of energy. The system can be written as a planar system as

$$\dot{\theta} = \phi$$

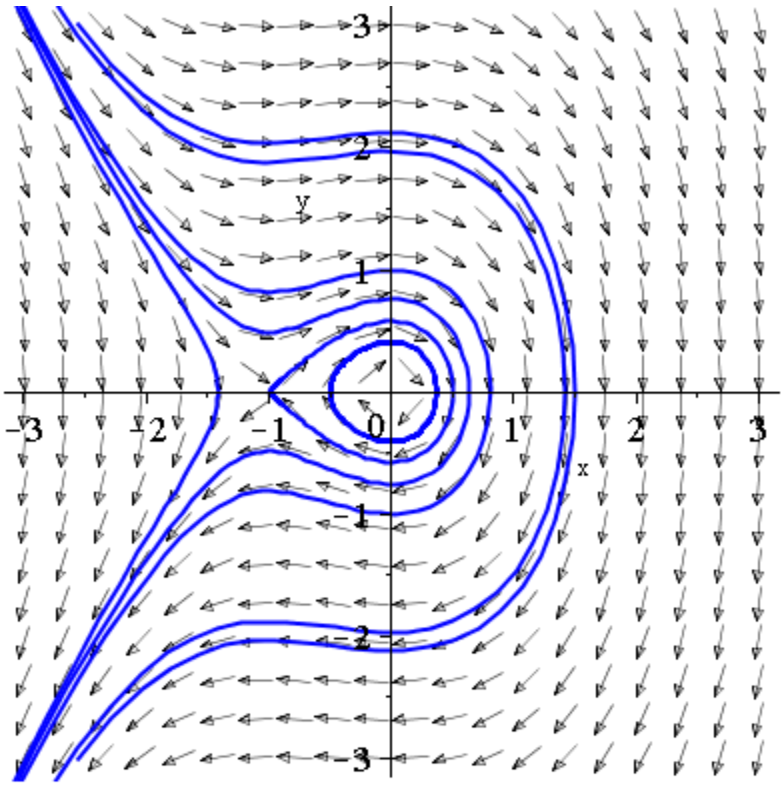
$$\dot{\phi} = -\frac{g}{l}\sin\theta$$

The system is a Hamiltonian system with

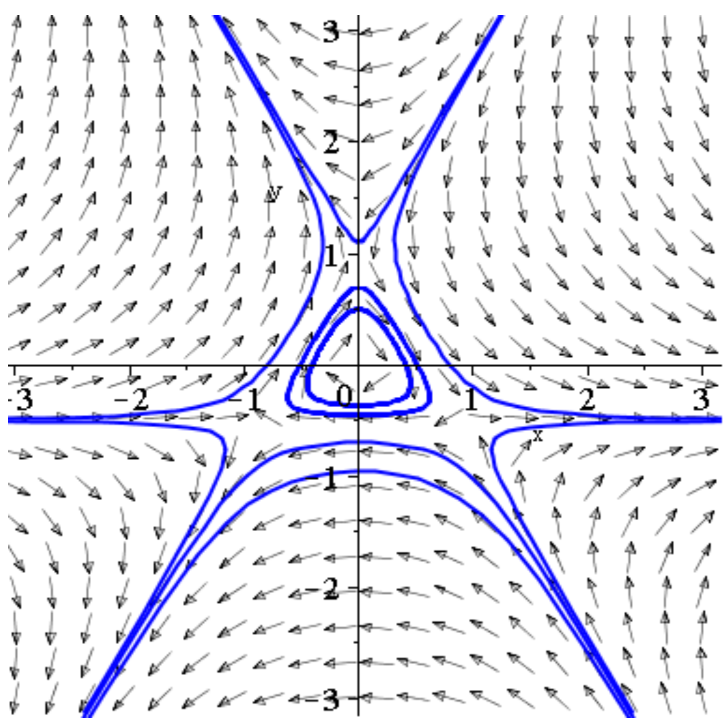
$$H(\theta, \phi) = \frac{\phi^2}{2} - \frac{g}{l}\cos\theta$$



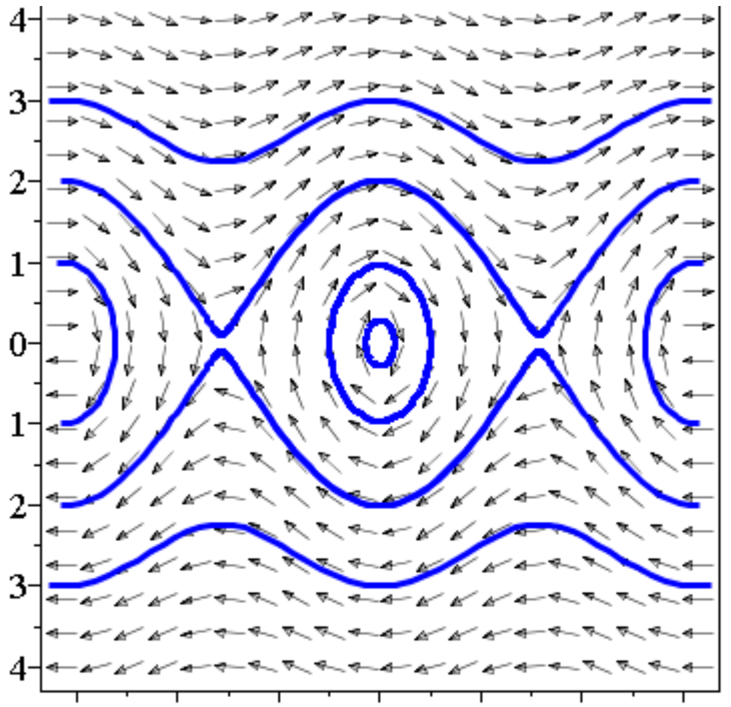
Phase portrait of example 7(a)



Phase portrait of example 7(b)



Phase portrait of example 7(c)



Example 8

Definition: Suppose $\mathbf{x}^*$ is a critical point of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2.$$

If $\gamma_+(\phi(t, \mathbf{x}_0)) = \gamma_-(\phi(t, \mathbf{x}_0)) = \mathbf{x}^*$, then the trajectory is a *homoclinic* orbit.

Definition: Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are distinct critical points. If $\gamma_+(\phi) = \mathbf{x_1}$ and $\gamma_-(\phi) = \mathbf{x_2}$, then $\phi(t, \mathbf{x_0})$ is called a *heteroclinic* orbit.

1.4 Lyapunov Functions and Stability

If a critical point is nonhyperbolic, then a method due Lyapunov may sometimes be used to determine the stability of the critical point

The Lyapunov Stability Theorem: Given a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

Let E be an open subset of $\mathbb{R}^n$containing an isolated critical point $\mathbf{x}^*$. Suppose that $\mathbf{f}$ is continuously differentiable and that there exists a continuously function say $V(\mathbf{x})$, which satisfies the following conditions:

- $V(\mathbf{x}^*) = 0;$
- $V(\mathbf{x}) > 0$, if $\mathbf{x} \neq \mathbf{x}^*$,

where $\mathbf{x} \in E$. Then

(a) if $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in E$, $\mathbf{x}^*$ is stable.
(b) if $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in E$, $\mathbf{x}^*$ is asymptotically stable.
(c) if $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \in E$, $\mathbf{x}^*$ is unstable.

Definition: The function $V(\mathbf{x})$ **is** called a Lyapunov function**.**

Unfortunately, there is no systematic way to construct a Lyapunov function. Note that if $\dot{V}(\mathbf{x}) = 0$, then all trajectories lie on the curves ("surfaces" in $\mathbb{R}^n$) defined by $V(\mathbf{x}) = C$, where C is a constant. The quantity $\dot{V}$ gives the rate of change of V along trajectories; in other words, $\dot{V}$ gives the direction that the trajectories cross the level curve $V(\mathbf{x}) = C$.

Definition: Given a Lyapunov function $V(x, y)$, the Lyapunov domain of stability is defined by the region for which $\dot{V}(x, y) < 0$.

Definition: A set S is called *positively invariant* provided that, whenever $\boldsymbol{x}_0$ is in S, then $\phi(t;\mathbf{x}_0)$ is in S for all $t \geq 0$.

Examples:

(9) Investigate the stability of the origin for the system

$$\dot{x} = y$$

$$\dot{y} = -x - y(1 - x^2)$$

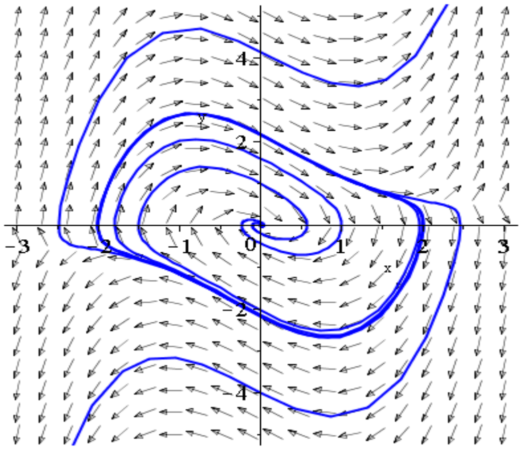
Using the Lyapunov function $V(x, y) = x^2 + y^2$.

(10) Investigate the stability of the origin for the system

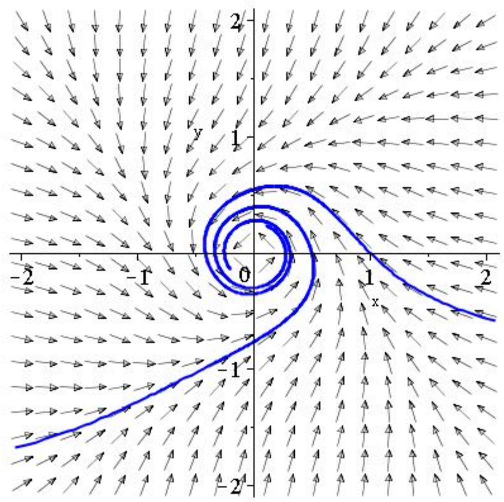
$$\dot{x} = -y - x^3$$

$$\dot{y} = x - y^3$$

Using the Lyapunov function $V(x, y) = x^2 + y^2$.



Example 9



Example 10

1.5 The Routh-Hurwitz Criteria

Given

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

The jacobian

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \frac{\partial f_n}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

The characteristic equation is of the form

$$\det(\mathbf{J} - \lambda\mathbf{I}) = 0.$$

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \cdots + a_n = 0.$$

Define *n* matrices as follows:

$$\mathbf{H}_1 = (a_1) \qquad \mathbf{H}_2 = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} \quad \mathbf{H}_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$\mathbf{H}_j = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ a_{2j-1} & a_{2j-2} & a_{2j-3} & & \cdots & a_j \end{pmatrix} \ldots\ldots \mathbf{H}_k = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & a_k \end{pmatrix}$$

where the (l, m) term in the matrix $\mathbf{H}_j$ is

a_{2l-m}	for $0 < 2l - m < k$
1	for $2l = m$
0	For $2l < m$ or $2l > k + m$

Then all eigenvalues have negative real parts: that is, the critical point $\mathbf{x}^*$ is stable if and only if the determinants of all Hurwitz matrices are positive:

$$\det \quad \mathbf{H}_j > 0 \quad (j = 1,2,3,\dots,k)$$

Routh-Hurwitz Criteria for $k = 2,3,4$

$k = 2$	$a_1 > 0,\ a_2 > 0$	
$k = 3$	$a_1 > 0,\ a_3 > 0$	$a_1 a_2 > a_3$
$k = 4$	$a_1 > 0, a_3 > 0, a_4 > 0$	$a_1 a_1 a_1 > a_3^2 + a_1^2 a_4$

Example:

Given a two preys and one predator equations

$$\frac{dx}{dt} = \alpha xz + \beta xy - \gamma x,$$

$$\frac{dy}{dt} = \delta y - \epsilon xy,$$

$$\frac{dz}{dt} = \mu z(\nu - z) - \chi(xz)$$