You may know these.....

GRAPH OF FUNCTIONS

$y = \sin x$

$y = \ln x$

quadratic

$y = x^2$

cubic

$y = x^3$

reciprocal:

$y = \frac{1}{x}$

$y = e^x$

$y = x$

$y = 2x + 1$

$y = 3$

Circle:

$y^2 + x^2 = a^2$

$y = \sqrt{a^2 - x^2}$

$y = -\sqrt{a^2 - x^2}$

$x = \sqrt{a^2 - y^2}$

$x = -\sqrt{a^2 - y^2}$

PARABOLA:

$x = y^2$

$y = \sqrt{x}$

$y = -\sqrt{x}$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Chapter 1: Multivariables Functions

1.1 Functions of Two Variables

1.1.1 Function representations

1.1.2 3-D Coordinate System

1.1.3 Graph of two variable functions

1.1.4 Sketching of the function (3-D

*Level Curves

1.1.5 Domain and Range

1.2 Functions of Three Variables

1.2.1 Domain and Range

1.2.2 Level Surfaces

1.1 Functions of Two Variables

$$\boxed{z = f(x, y)}$$

Means that z is a function of x and y in the sense that a unique value of the **dependent variable** z is determined by specifying values for the **independent variables** x and y.

$$(x, y) \in Domain$$

$$z \in Range$$

and

x and y : the two different independent variables

z : the dependent variable

Domain (D) : the set of all possible inputs (x, y) of the function $f(x, y)$ that is

Range (R) : the set of output z that result when (x, y) varies over the domain D

For example,

1. $f(x, y) = \sqrt{x^2 + y^2}$

Function of two variables

$$f(1,1) = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Substitute 1 for x and 1 for y

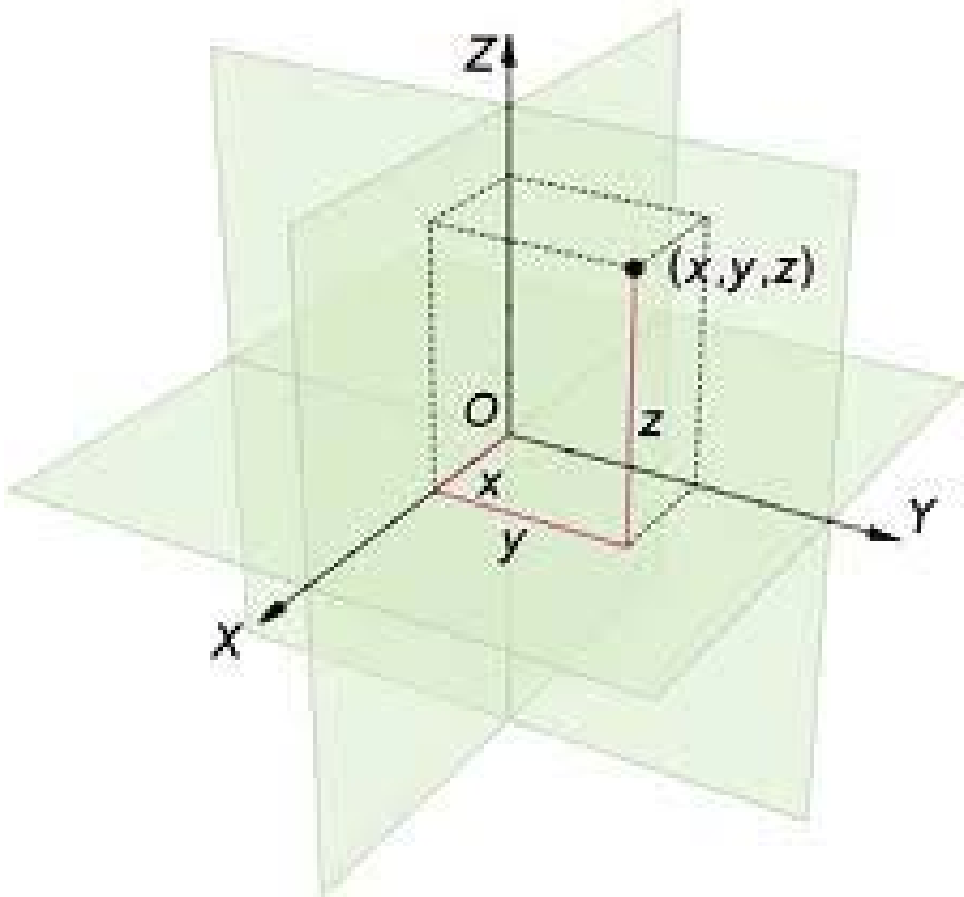
2. $z = f(x, y) = \sqrt{64 - x^2 + e^{xy}}$

$$f(1,0) = \sqrt{64 - 1 + 1} = 8$$

$$f(2,-3) = \sqrt{64 - 4 + e^{-6}} = \sqrt{60 + e^{-6}}$$

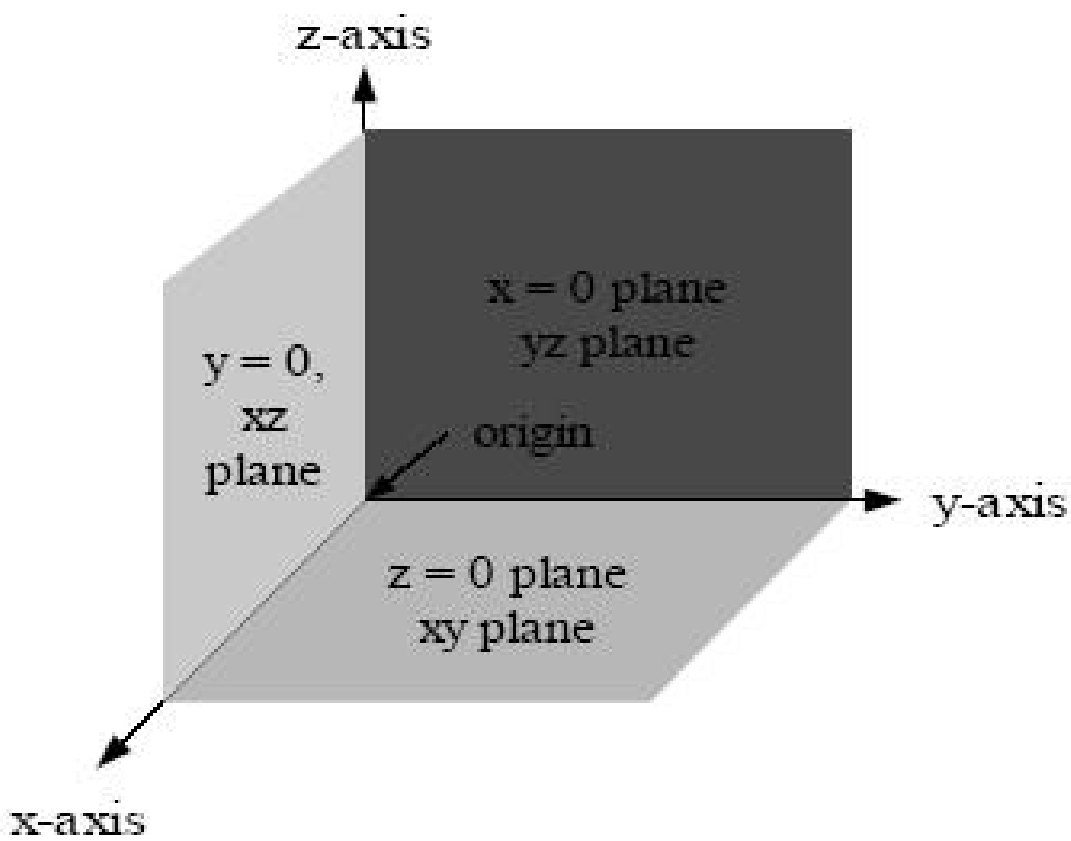
1.1.1 Function Representation of $z = f(x, y)$

3-D coordinate system

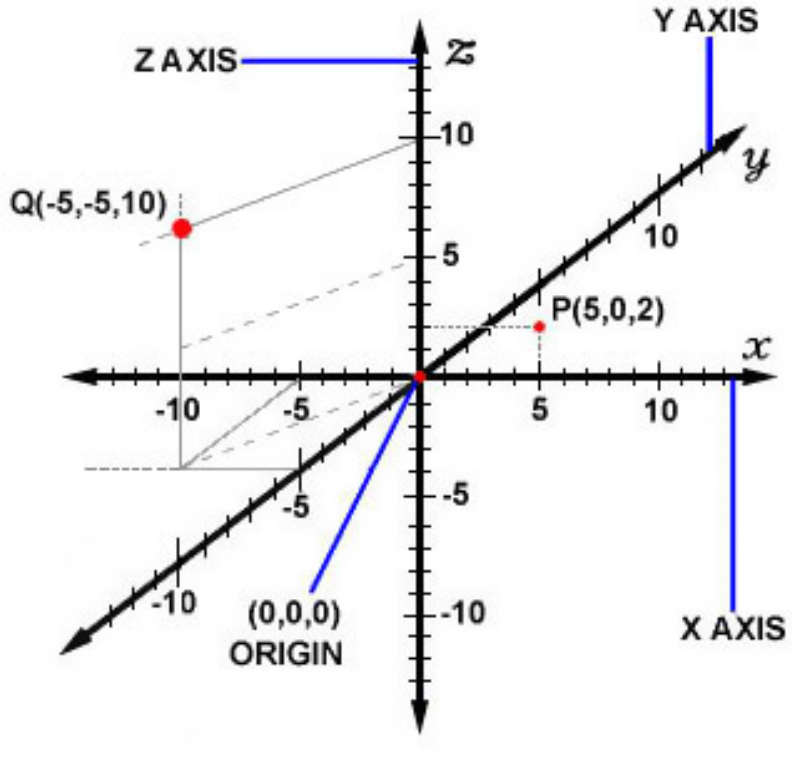


$f(x, y)$ is a rule that assigns a unique real number to each point *(x,y)* in same set *D* in the *xy*-plane

Coordinate Planes



1.1.2 3-D Coordinate system

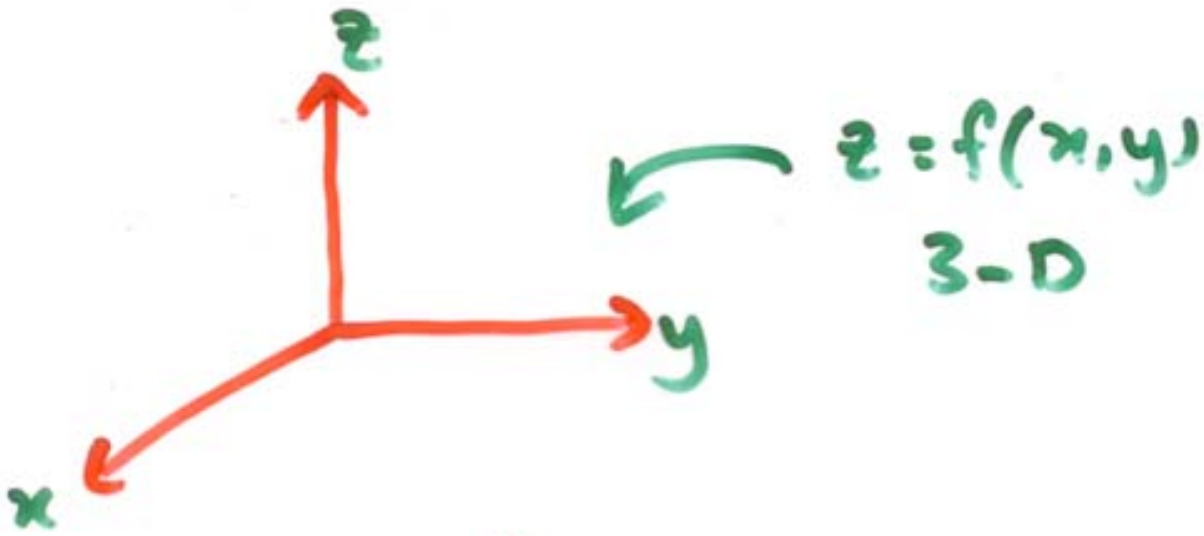


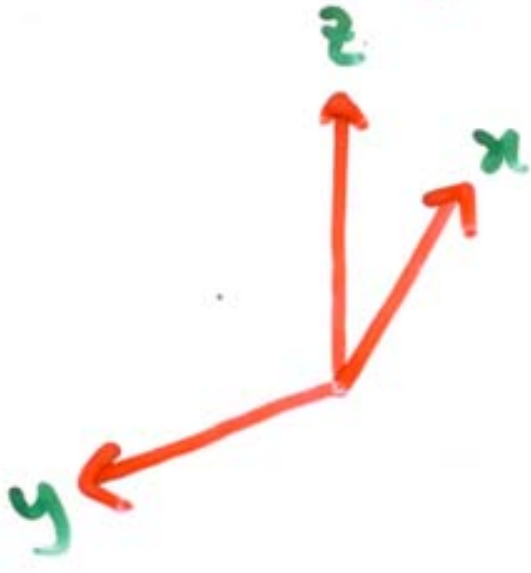
3 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

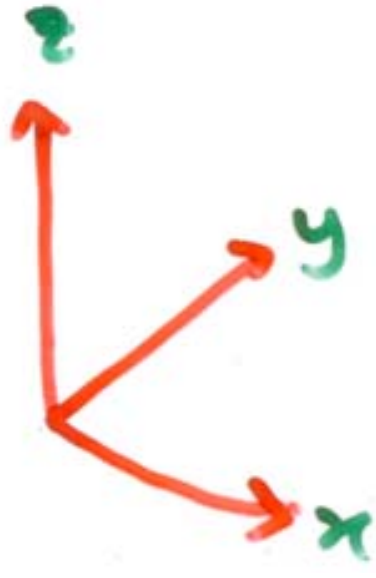
3D coordinate system has 3 main planes:-

xy plane	or	$z = 0$	$(x, y, 0)$
xz plane	or	$y = 0$	$(x, 0, z)$
yz plane	or	$x = 0$	$(0, y, z)$

The orientation of xyz-axis

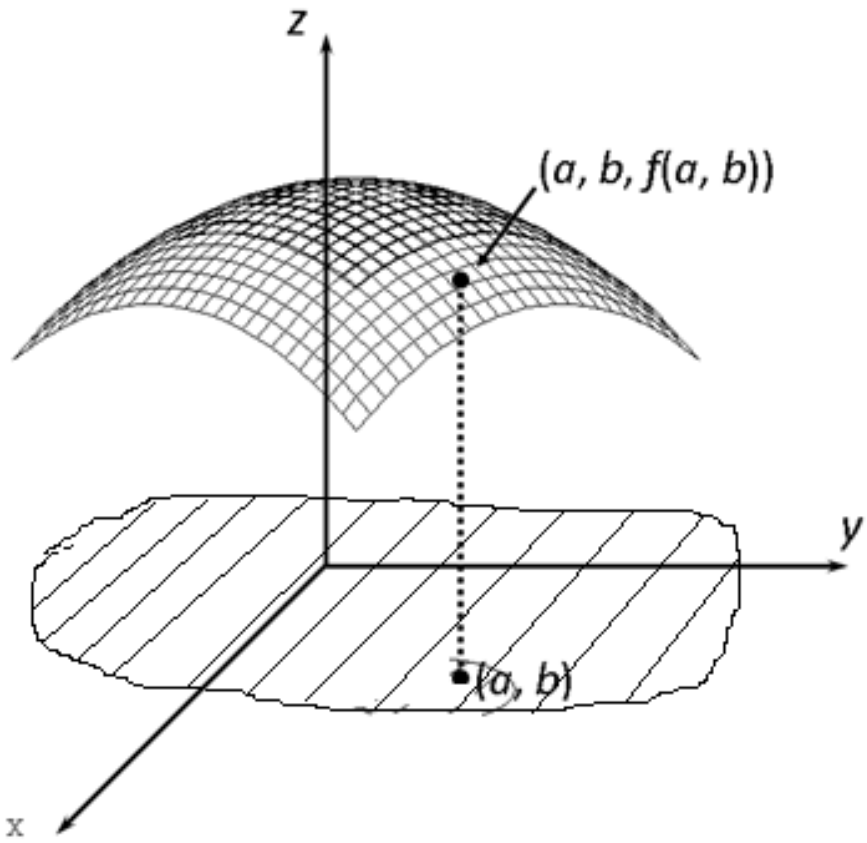


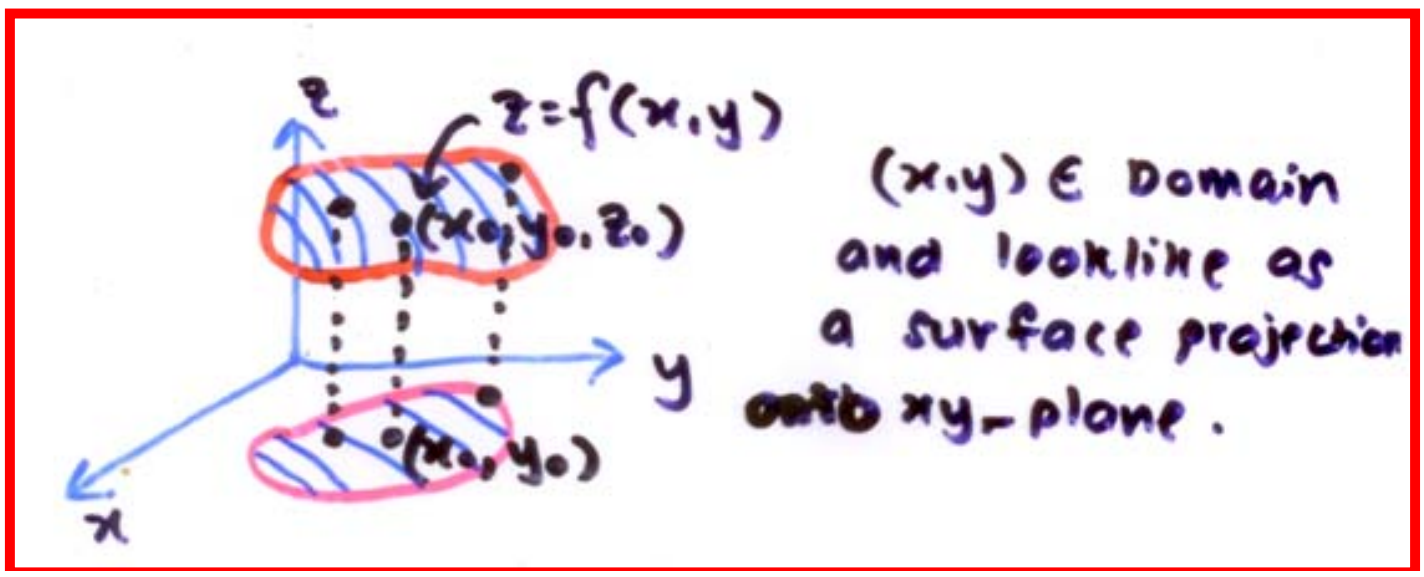




1.1.3 Graph of a Function of Two Variables

The **graph of the function *f* of two variables** is the set of all points (x, y, z) in three-dimensional space, where the values of (x, y) lie in the domain of f and $z = f(x, y)$.





The graphs of $z = f(x, y)$ is called a surface in 3D system or three-space ($\Re^3$).

It looks like a blanket!

Four types of surface in space:

1.1.3.1 Planes

Example 1

$z = 0, y = 0, x = 0$

$x = 3, y = -1, z = 5$

Given as a constant equation with **one-variable.**

Example 2

$y = -x + 6,\ 2y = 4z + 5,\ z + x = 4$

Given as a linear equation with **two-variable.**

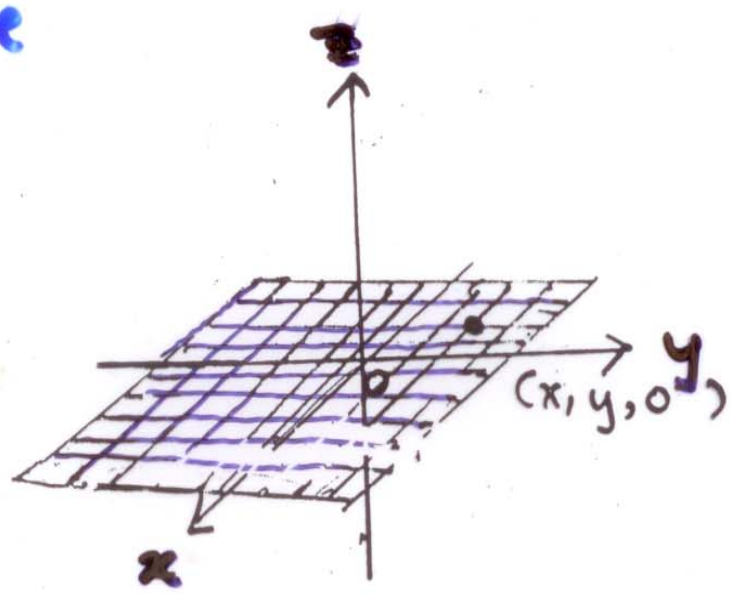
Example 3 Tetrahedron

$$y + x + y = 1$$

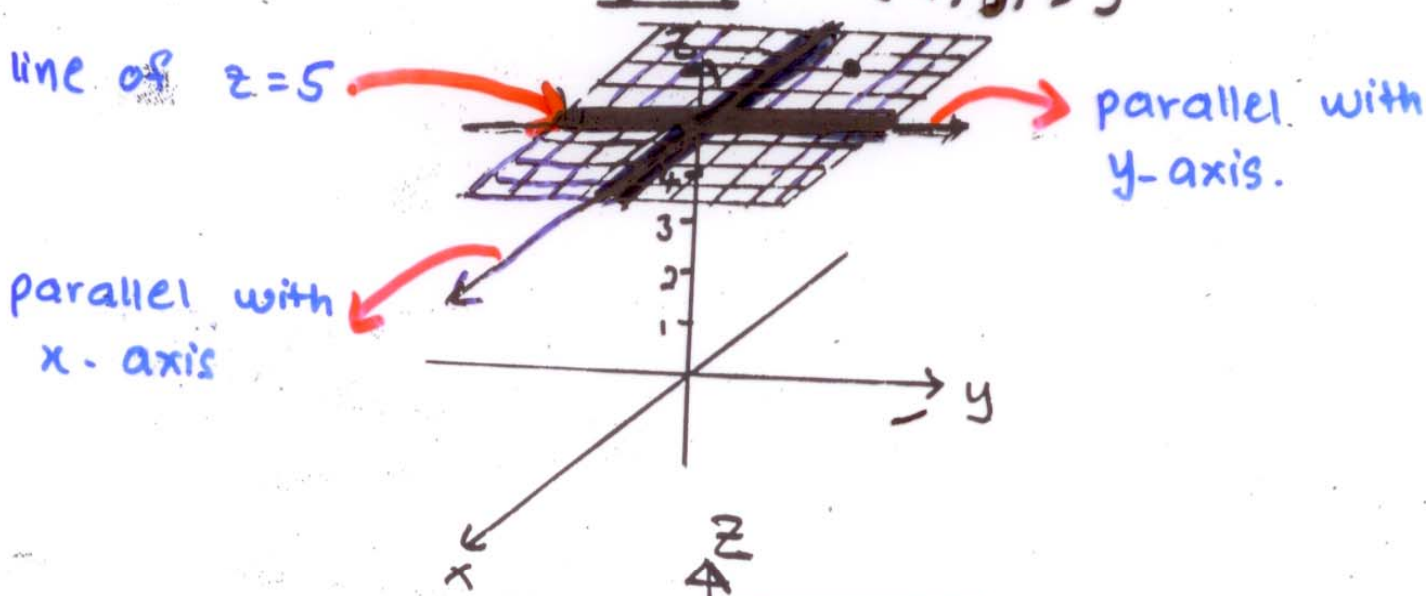
$$z = 6 - 3y + 2x$$

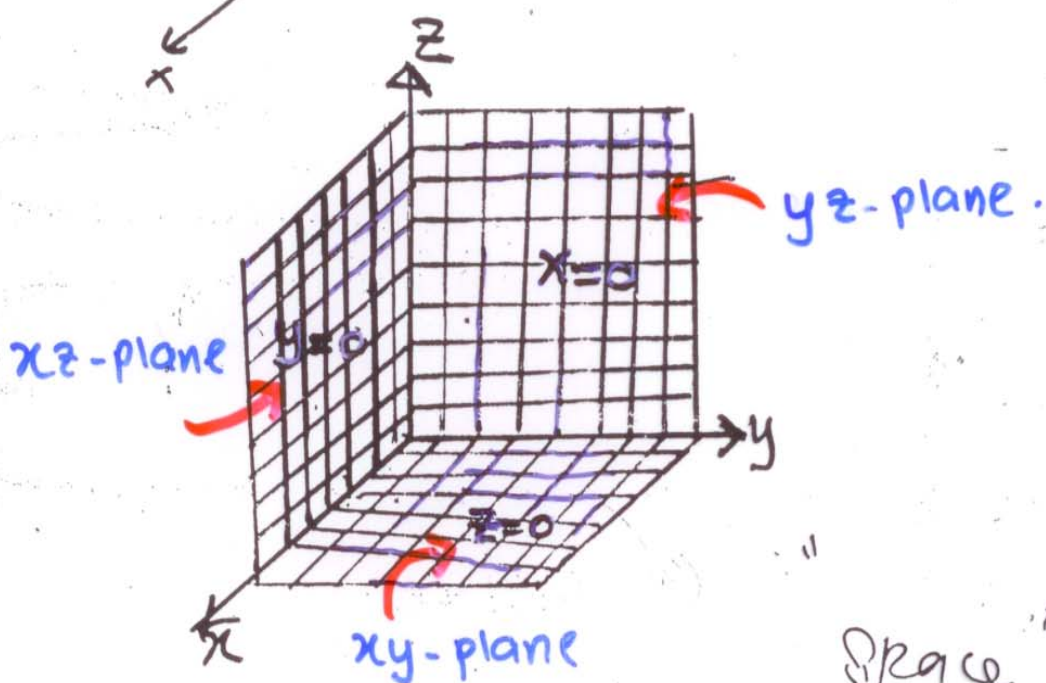
Given as a linear equation with **three-variable.**

xy-plane is called $z = 0$



$z = 5$ $(x, y, 5)$





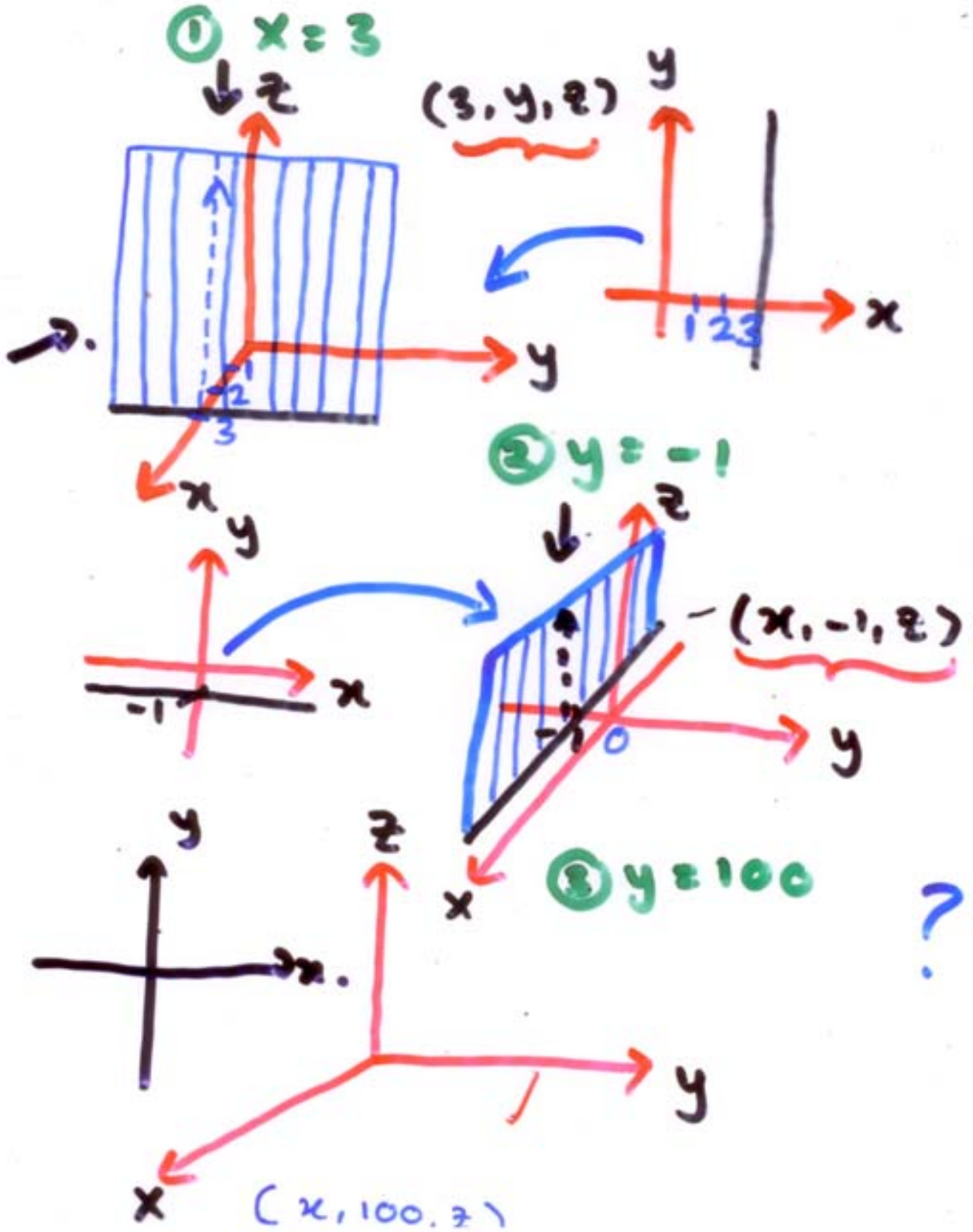
"Space"

"ruang"

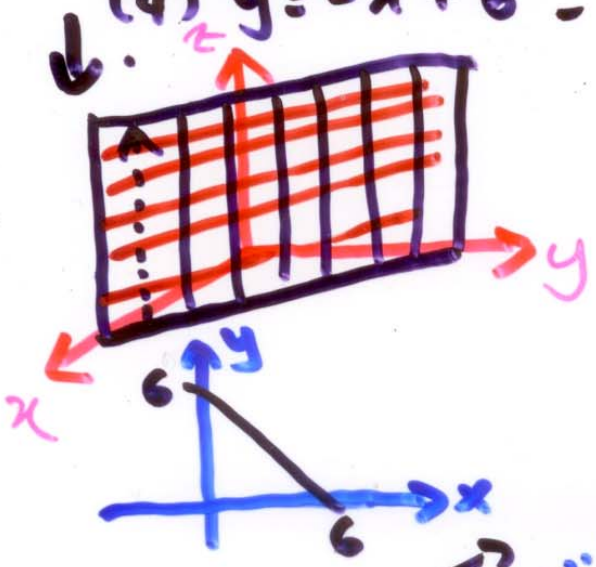
How to sketch of the given functions

1) **Determine the variables**
2) **Sketch the trace in coordinate planes (based on the variables exist)**
3) **Make the projection onto the trace-plane which is parallel to the (variables which is not exists)-axis**

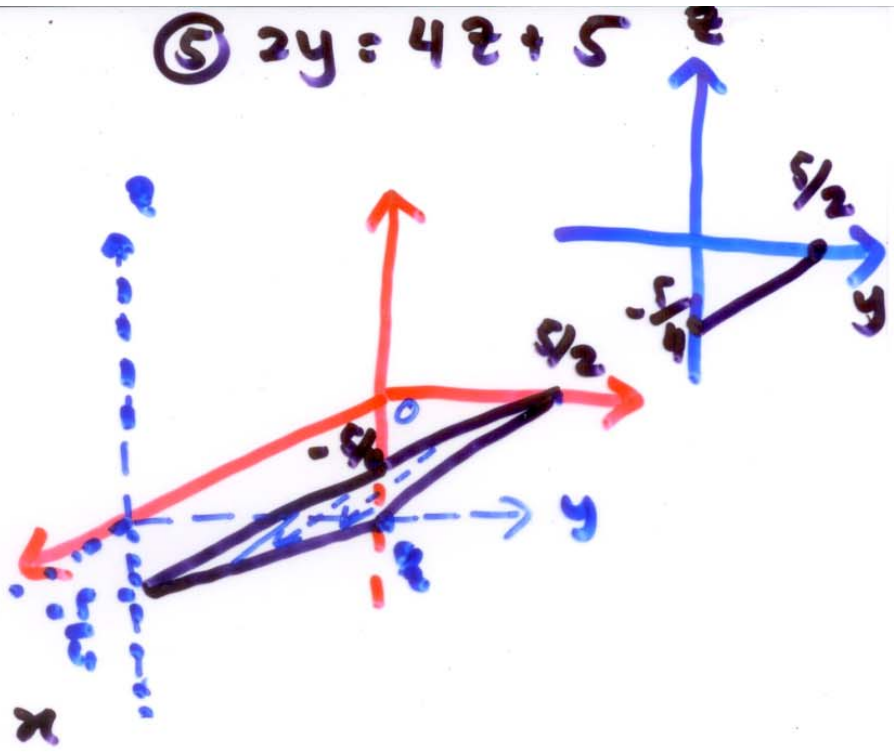
- Sketch the trace in the xy-plane.
- Then, the projection onto xy-plane is called the plane which parallel to yz-plane



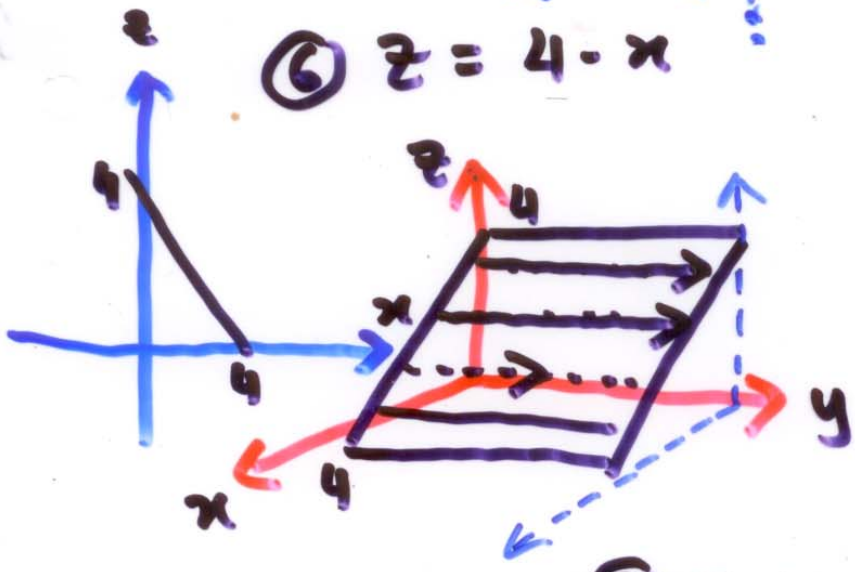
(4) $y = -x + 6$



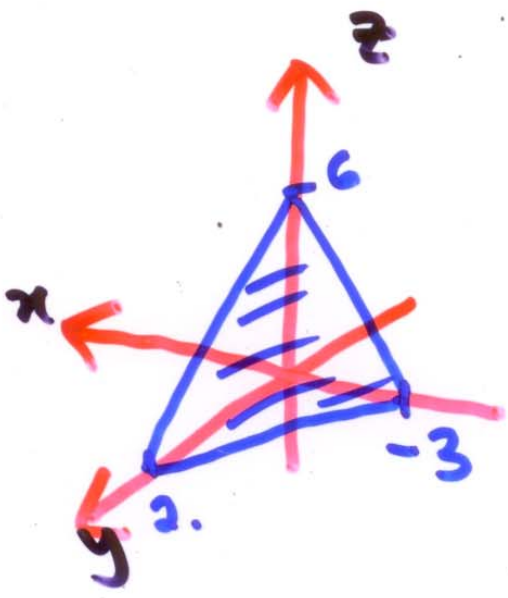
⑤ $2y = 4z + 5$



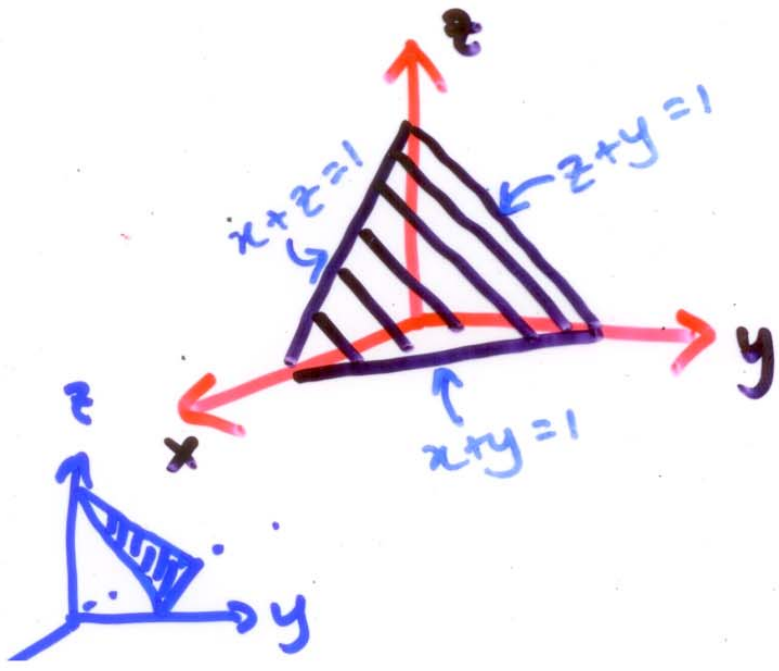
⑥ $z = 4 - x$



⑧ $z = 6 - 3y + 2x$



⑦ $z + x + y = 1$



Eg 7 : Sketch the graph of $z + x + y = 1$.

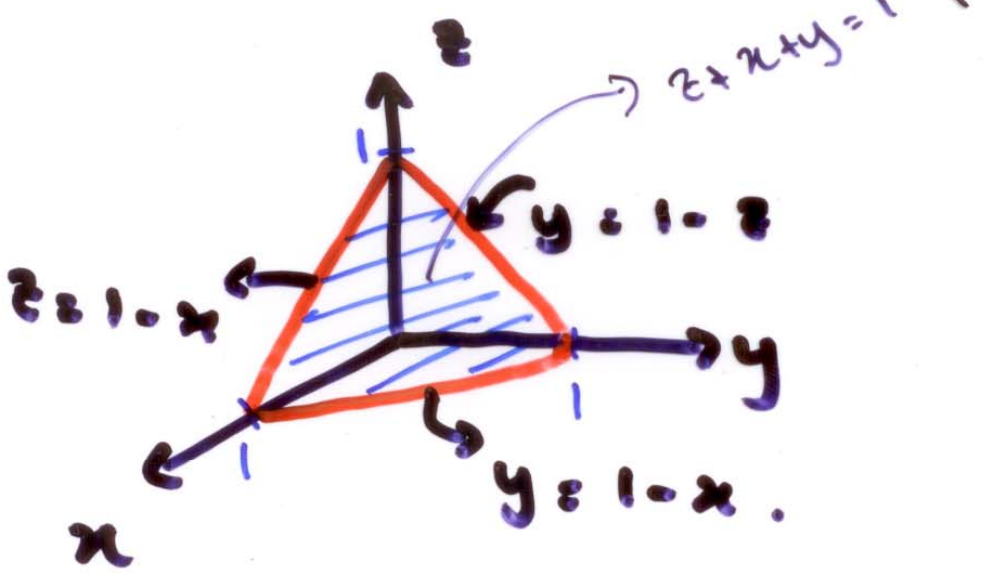
Solution :

The traces in the coordinate planes :

→ yz - plane , $x = 0$: the straight line $y = 1 - z$.

→ xz - plane , $y = 0$: the straight line $z = 1 - x$.

→ xy - plane , $z = 0$: the straight line $y = 1 - x$.



Exercise 1 :

Sketch the graph of

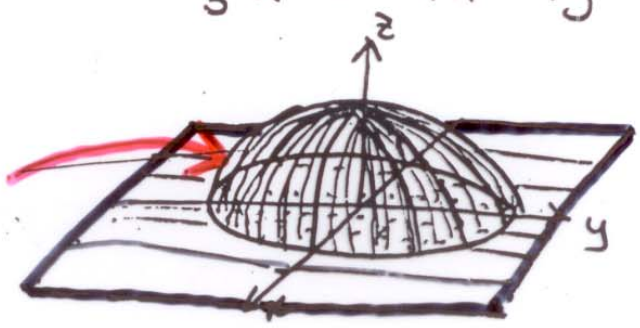
$z = 6 - 3y + 2x$

1.1.3.2 Curved surfaces

Examples of the graph $z = f(x,y)$

(i) $z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$

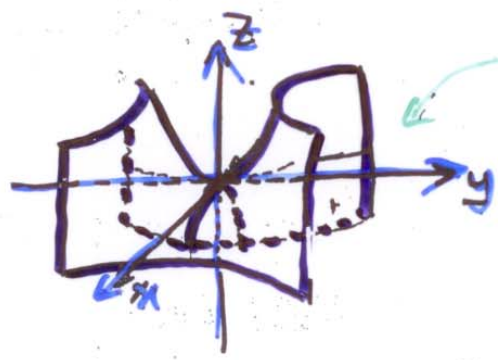
hemisphere



(ii) $z = -4x^2y^2$

pelana (saddle)

(iii) $z = y^2 - x^2$



pelana

Seperti "saddle" kuda.

1.1.3.2 CURVED SURFACE
(NONLINEAR EQUATION)

① Eg: $y = x^2$

$x^2 + y^2 = 25$

$y^2 - z^2 = 9$

- the given eq^{n}s have two variables.
- firstly, sketch the graph trace in a plane (based on the given variables)

② Eg: $z = x^2 + y^2$

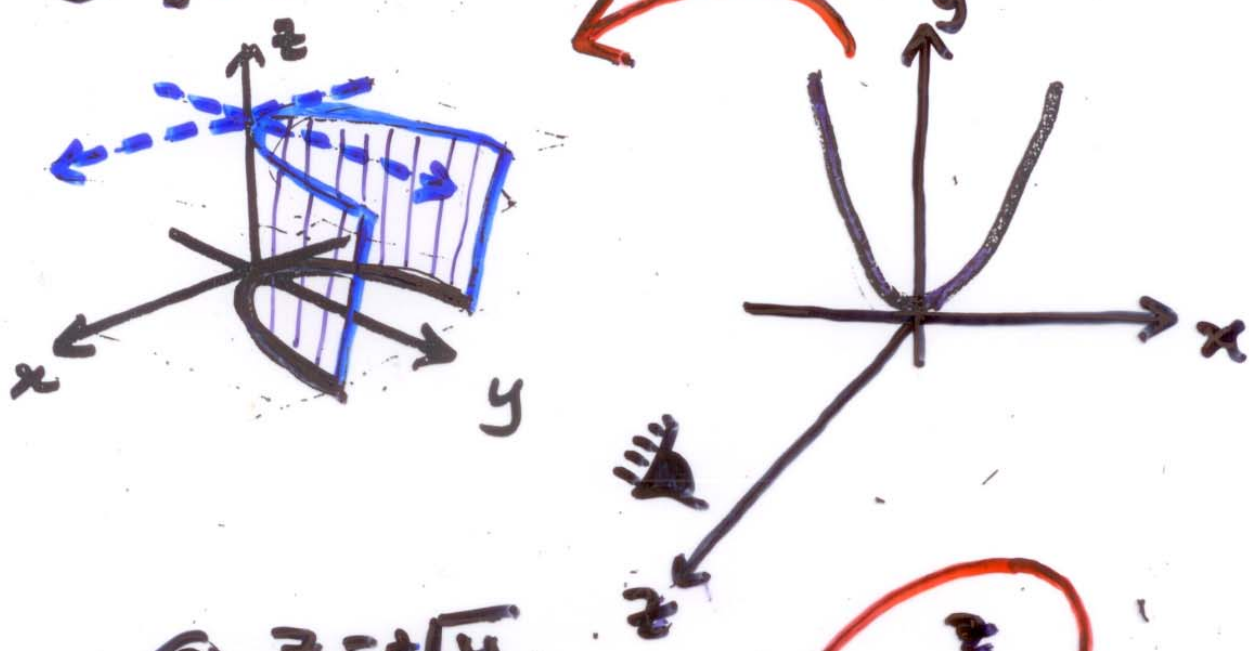
$z = \sqrt{x^2 + y^2}$

$\frac{x^2}{9} + \frac{y^2}{4} - z^2 = 1$

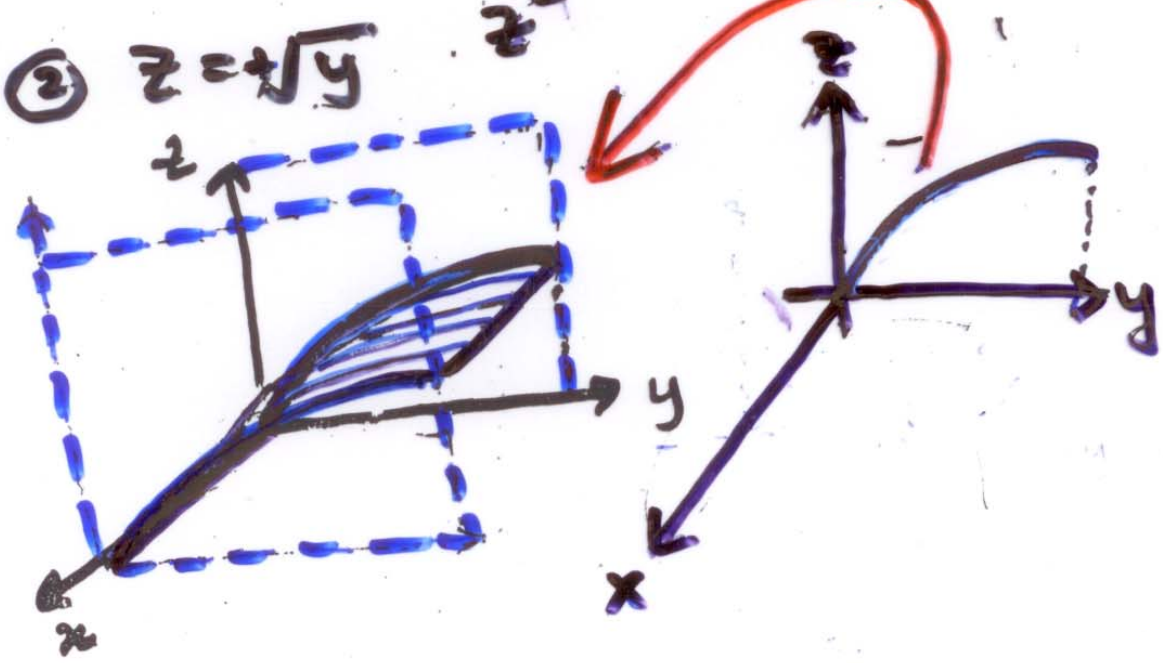
- the given eq^{n}s have three variables
- firstly, sketch the graph traces in the three planes (coordinate planes)

? ? ? ? ? ?

① $y = x^2$

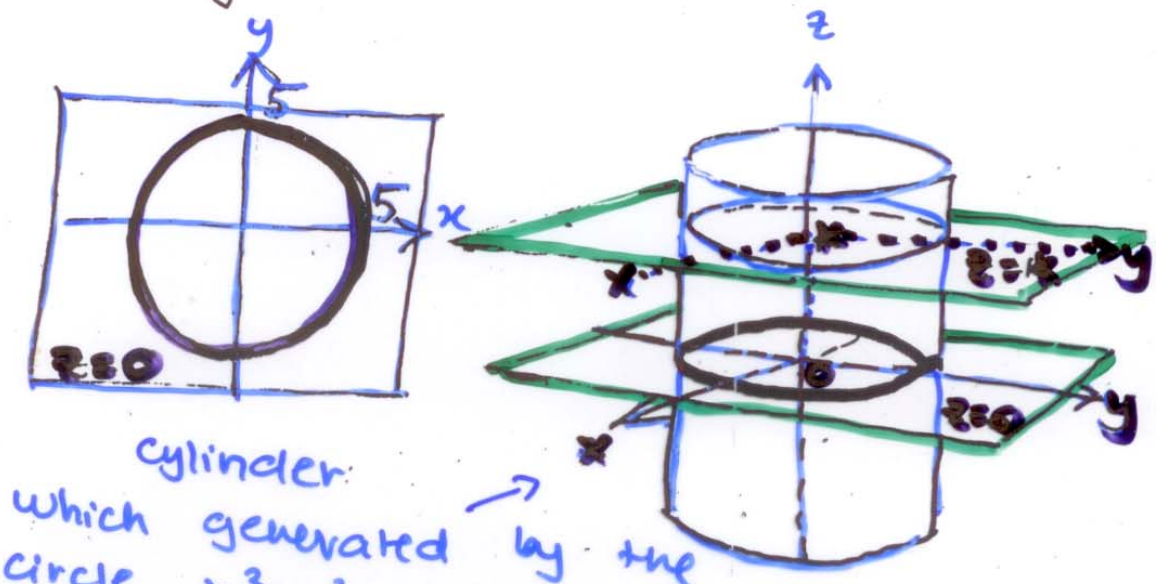


② $z = +\sqrt{y}$



Sketch the graph trace in a plane (based on the given variables)

③ $x^2+y^2=25$; circle

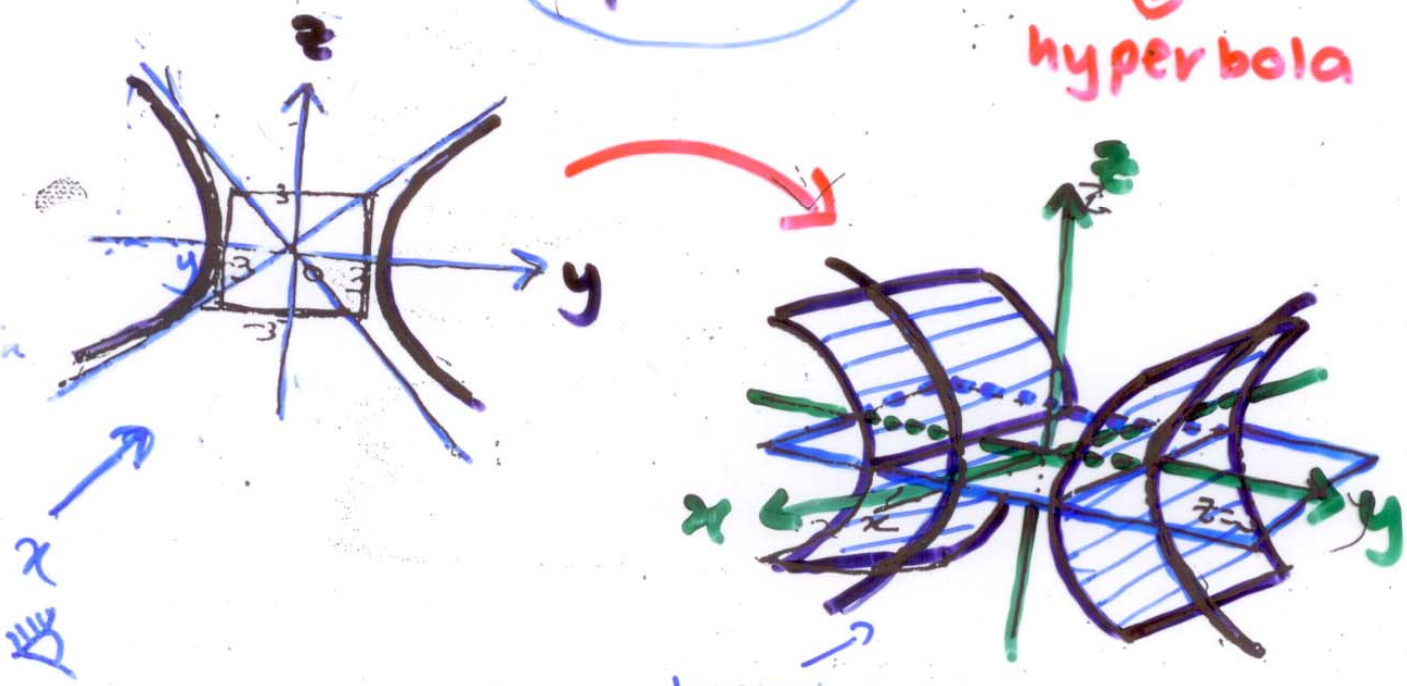


cylinder which generated by the circle $x^2+y^2=25$ and lies parallel to the z-axis.

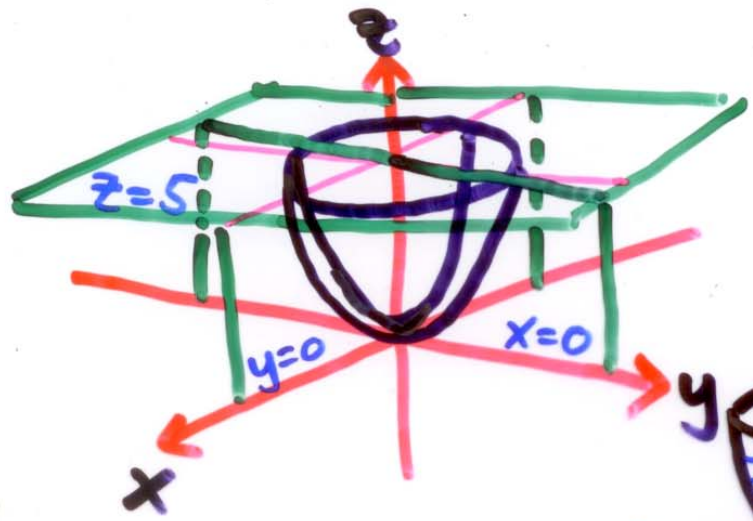
⑤ $y^2-z^2=9$

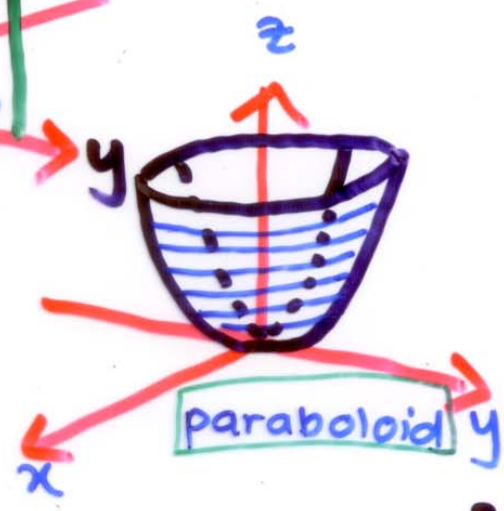
$\Rightarrow \frac{y^2}{9}-\frac{z^2}{9}=1 \Rightarrow \frac{y^2}{3^2}-\frac{z^2}{3^2}=1$

hyperbola



hyperboloid lies parallel to x-axis





① $z = x^2 + y^2$

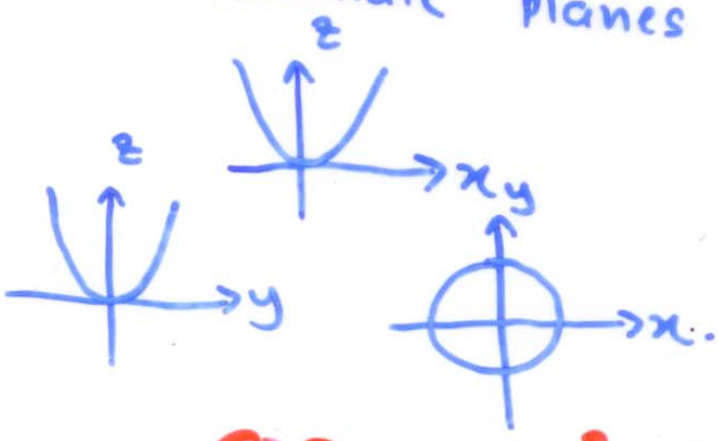
The graph traces in the coordinate planes

→ $y=0$: $z = x^2$

→ $x=0$: $z = y^2$

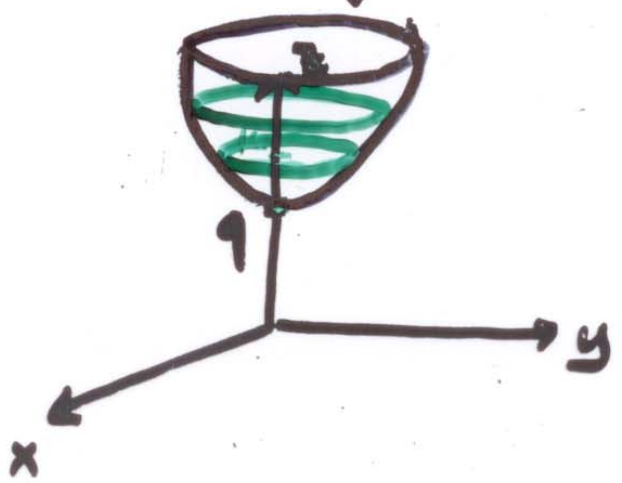
and $z=5$: $x^2+y^2=5$

circle of radius $\sqrt{5}$

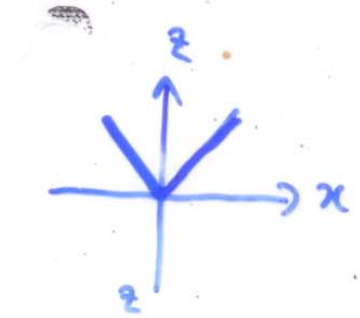


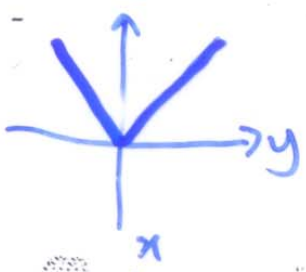
② $z = -(x^2+y^2)$

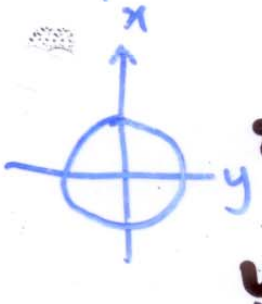
③ $z = x^2 + y^2 + 1$

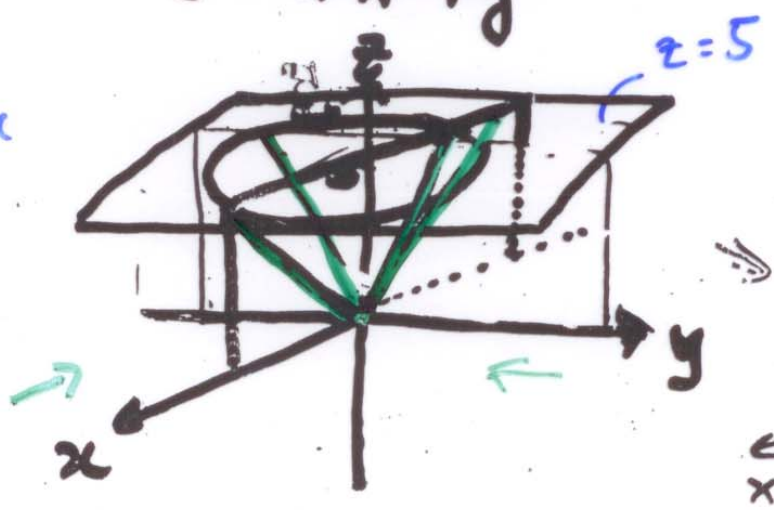


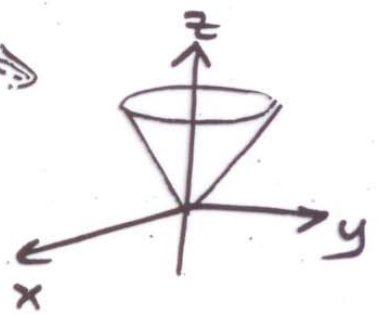
④ $z = \sqrt{x^2+y^2}$











xz-plane: $z = \sqrt{x^2}$ $\quad y=0$

$\Rightarrow z = x$ or $z = -x$

yz-plane: $= z = \sqrt{y^2}$, $x = 0$

$\Rightarrow z = y$ or $z = -y$

plane: $z = 5 \Rightarrow 5 = \sqrt{x^2+y^2}$

$\Rightarrow x^2 + y^2 = 25$

(5) $\frac{x^2}{3^2} + \frac{y^2}{2^2} - z^2 = 1$

trail of xy-plane : ; $z=0$

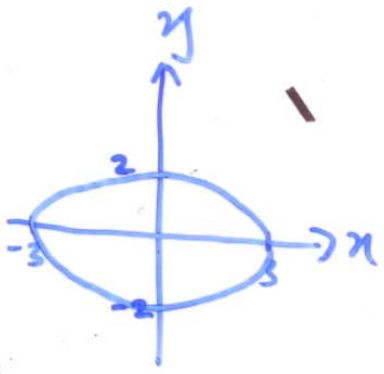
$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ ellipse

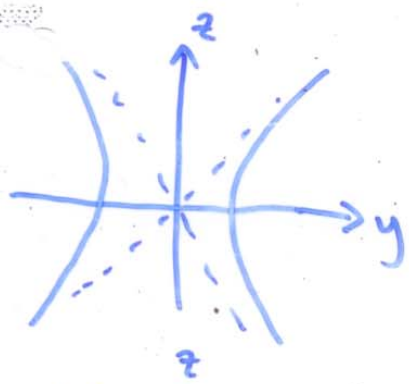
trail of yz-plane : $x=0$

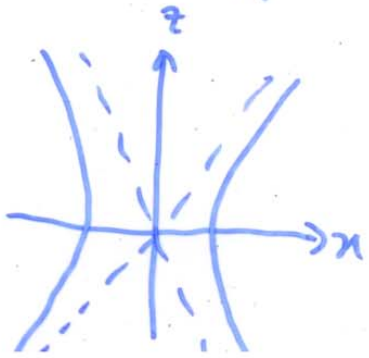
$\frac{y^2}{2^2} - z^2 = 1$ hyperbola

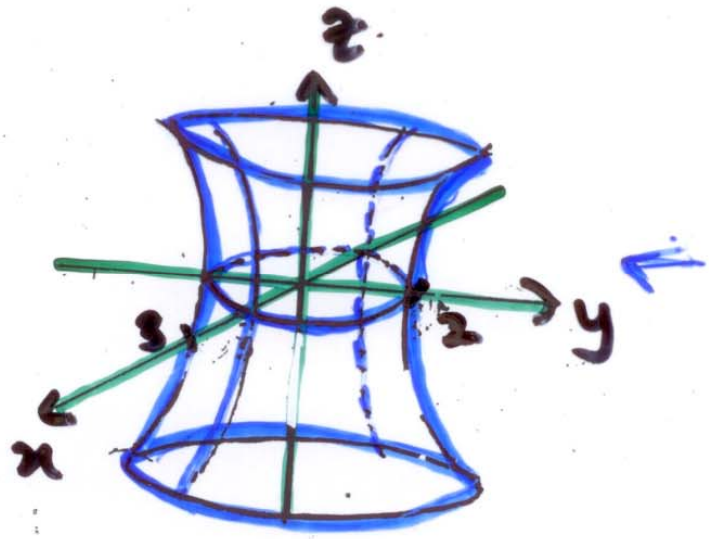
trail of xz-plane , $y=0$

$\frac{x^2}{3^2} - z^2 = 1$ hyperbola

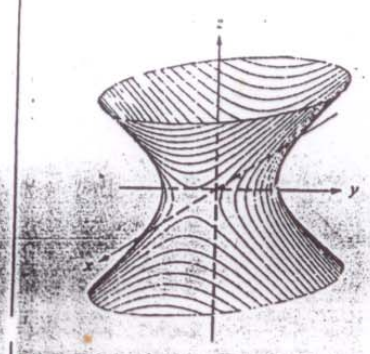








hyperboloid.



Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces parallel to xy-plane: Ellipses

Traces parallel to xz-plane: Hyperbolas

Traces parallel to yz-plane: Hyperbolas

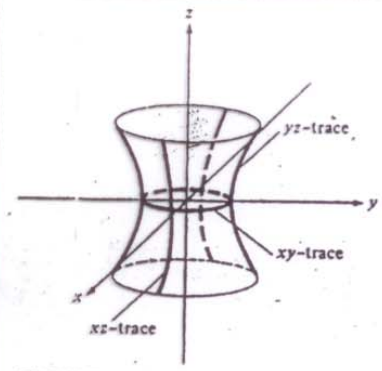
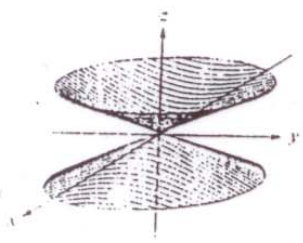


FIGURE 14.38



Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Traces parallel to xy-plane: Ellipses

Traces parallel to xz-plane: Hyperbolas

Traces parallel to yz-plane: Hyperbolas

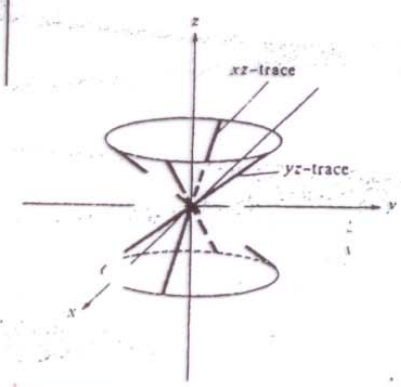
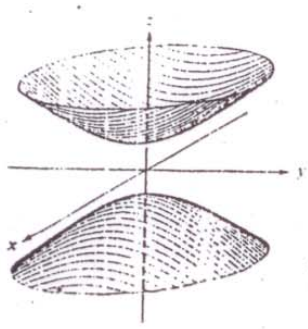


FIGURE 14.39



Hyperboloid of Two Sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

Traces parallel to xy-plane: Ellipses

Traces parallel to xz-plane: Hyperbolas

Traces parallel to yz-plane: Hyperbolas

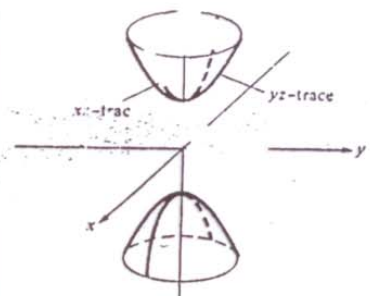


FIGURE 14.40

(14)

How to sketch curved surfaces ?

- ✓ domain and range
- ✓ Level curves

Level Curves

To sketch the graph of two variables, we need to familiar with the contour maps.

Notice that when the plane $z = C$ intersects with the surface $z = f(x, y)$, the result is the space curve with the equation $f(x,y)=C$, so we called these as the **level curves.**

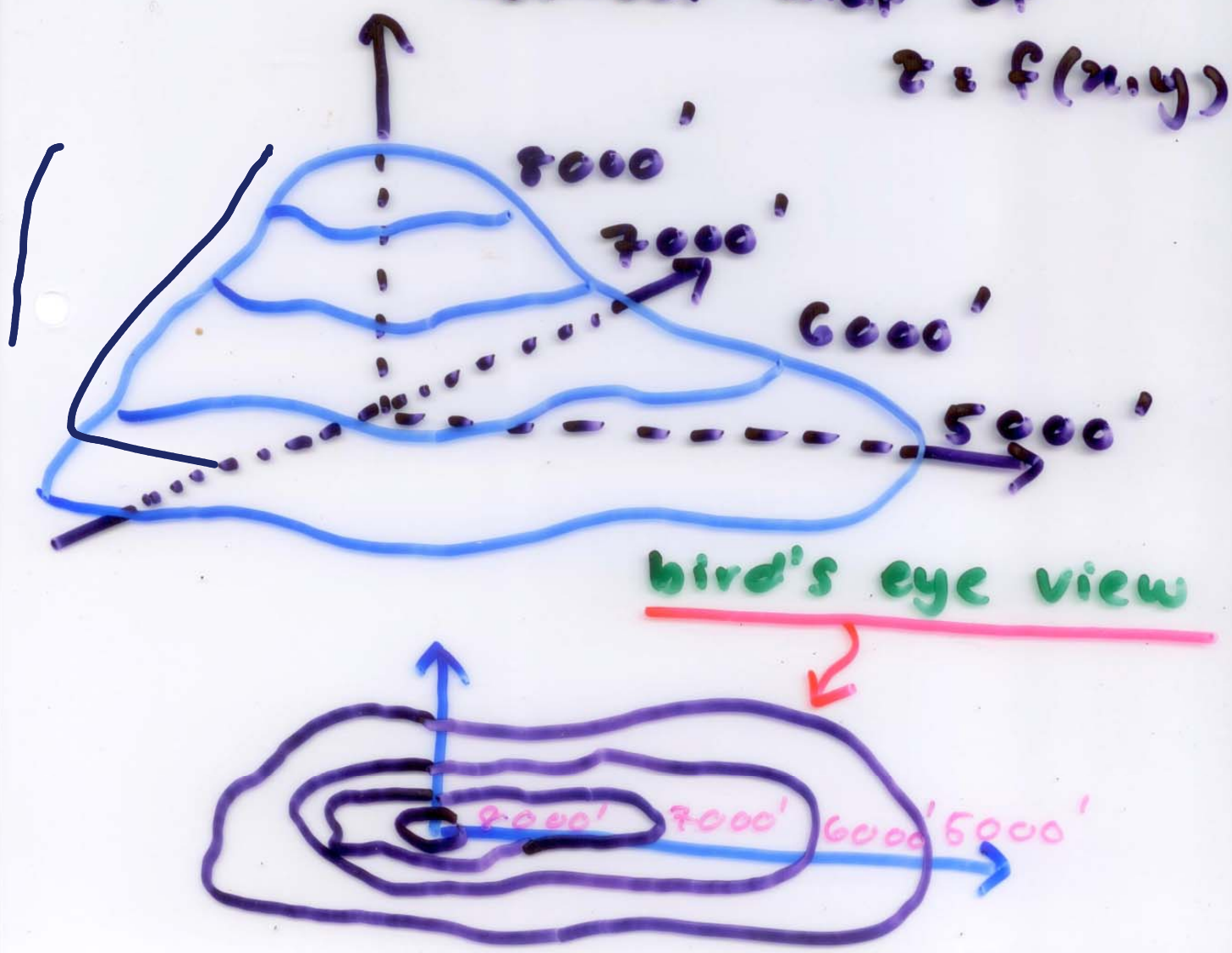
1.1.4 Sketch of the surface

$$z = f(x, y)$$

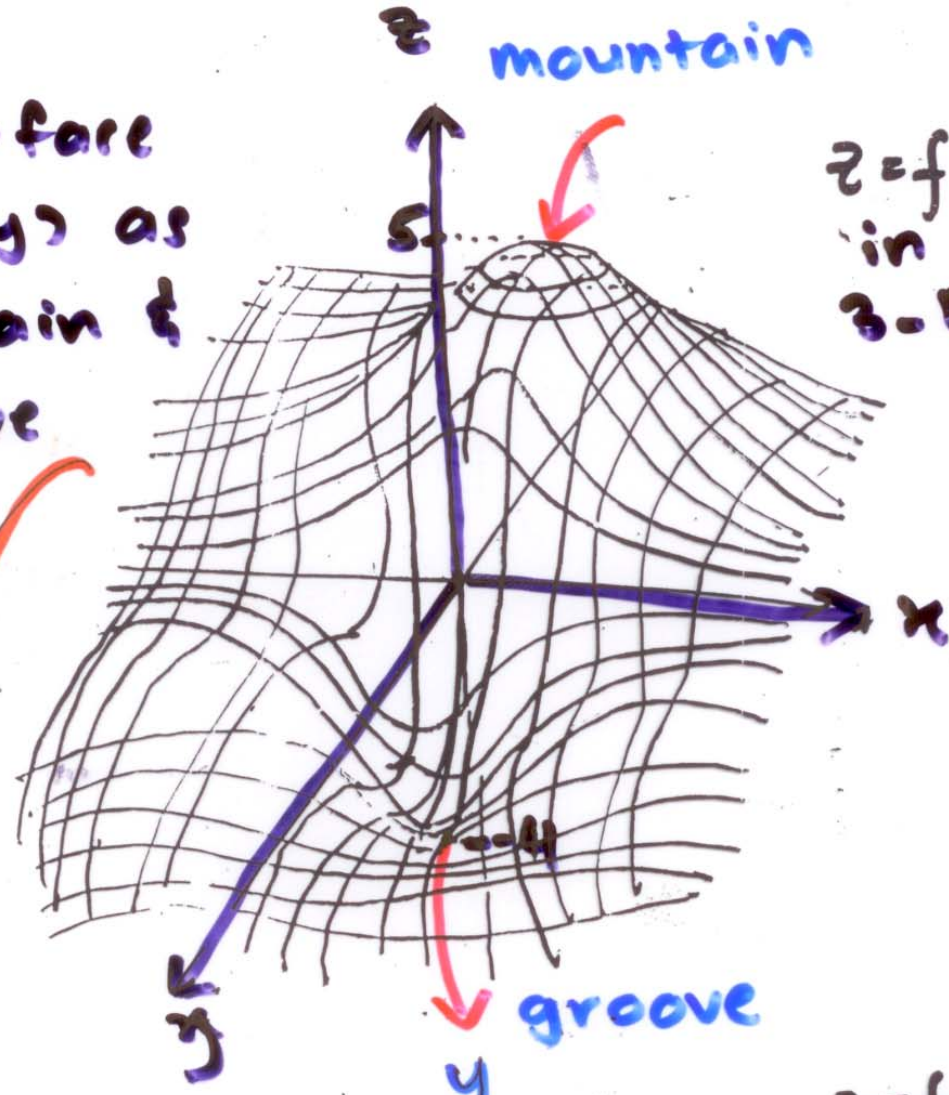
***** the set of point (x, y) in xy-plane that satisfy $f(x, y)$ is called **level curves /contour curves**

Eg: CONTOUR MAP

Level curves yield a contour map of $z = f(x,y)$

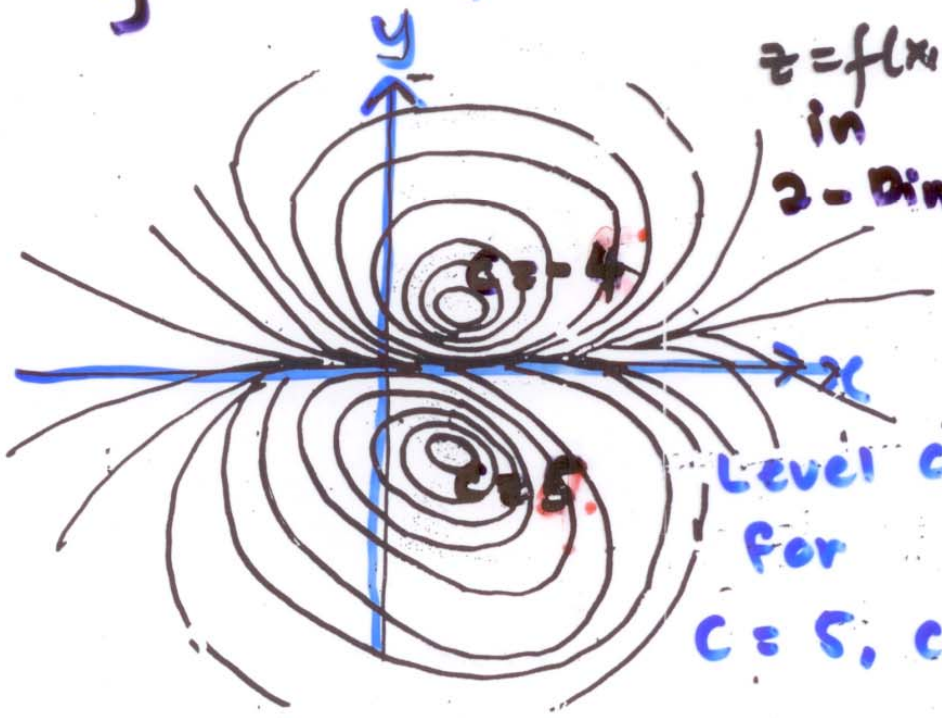


the surface $z = f(x,y)$ as a mountain & a groove



$z = f(x,y)$ in 3-Dimension

contour map



$z = f(x,y)$ in 2-Dimension

Level curves for $c = 5$, $c = -4$ etc...

Sketching surfaces with level curves

Let $z = f(x, y)$ is a function of two variables

- ✓ Plane $z = C$ intersects with the surface $z = f(x, y) \rightarrow f(x, y) = C$
- ✓ The set of point (x, y) in the xy-plane that satisfy $f(x, y) = C$ **is called the level curve of *f* at *C***
- ✓ An entire family of level curves is generated as C varies over the range of f
- ✓ The graph of $z = f(x, y)$ is a surface which can be obtained by sketching the contour map (set of level curves) on xy-plane

Example

Sketch the contour lines/level curves and the graphs

(i) $z = x^2 + y^2$, $c = 0, 1, 2, 3, 4, 9$

(ii) $z = \sqrt{x^2 + y^2}$, $c = 0, 1, 4, 9$

(iii) $z = 6 - x^2 - y$, $c = 0, 2, 4, 6$

Solution

(i) $z = x^2 + y^2$, $c = 0, 1, 2, 3, 4, 9$

Sketching the level curves

- first, replace z with the value of c

- second, plot the graph on the xy-plane

$c = 0 \quad : x^2 + y^2 = 0$

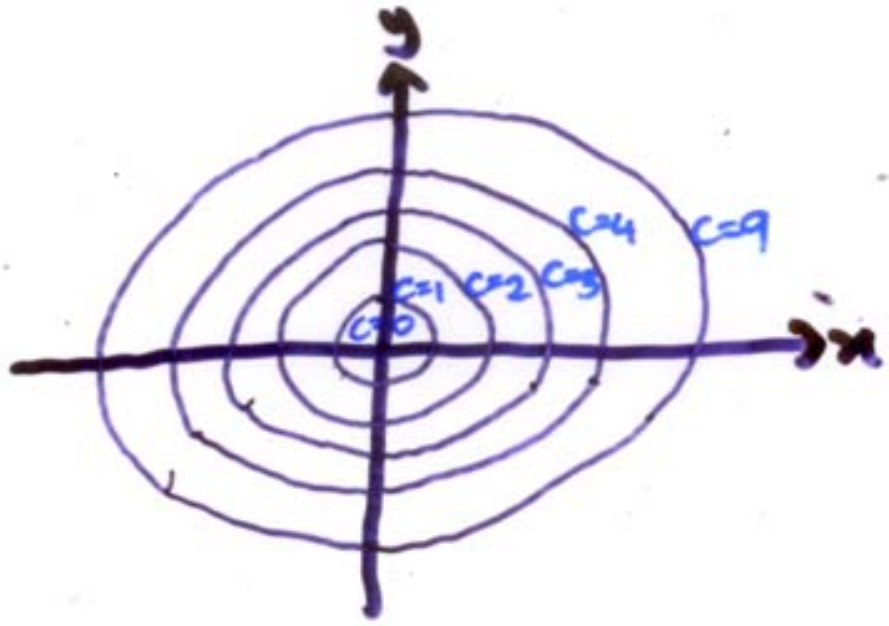
$c = 1 \quad : x^2 + y^2 = 1$

$c = 2 \quad : x^2 + y^2 = 2$

$c = 3 \quad : x^2 + y^2 = 3$

$c = 4 \quad : x^2 + y^2 = 4$

$c = 9 \quad : x^2 + y^2 = 9$

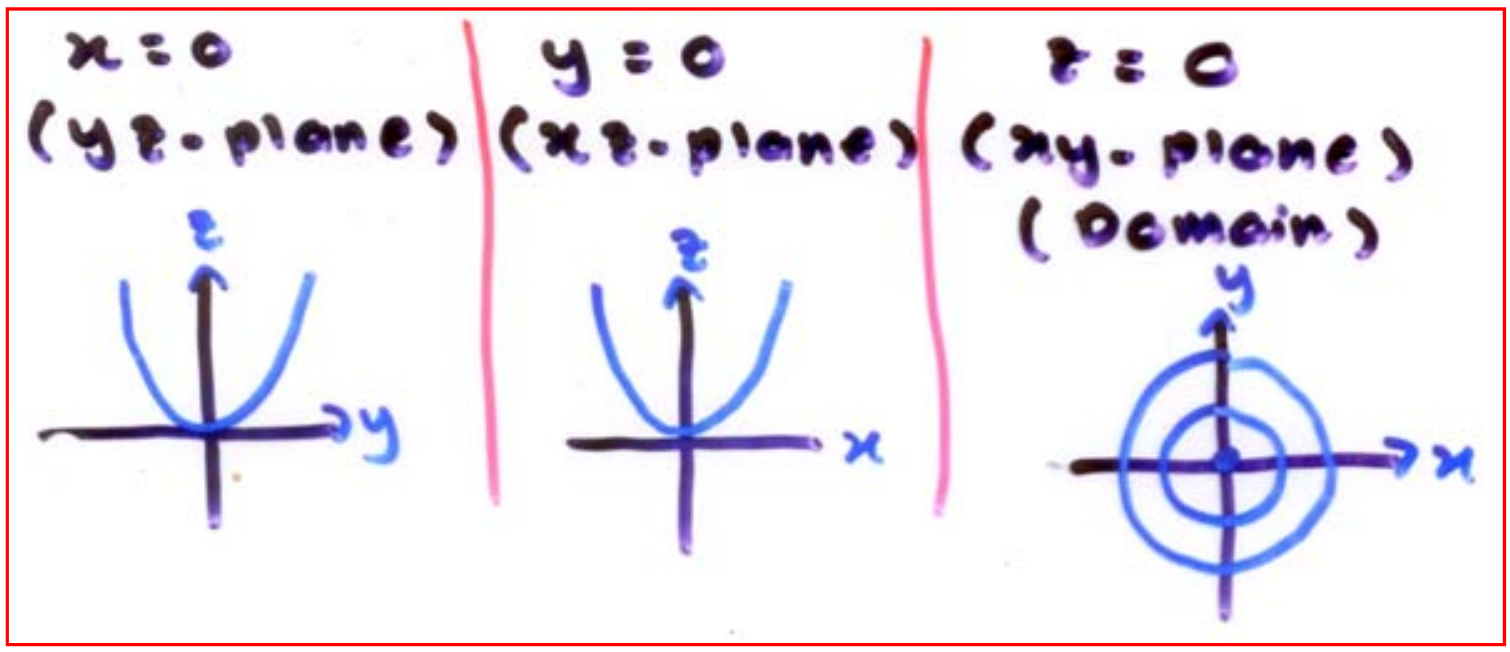


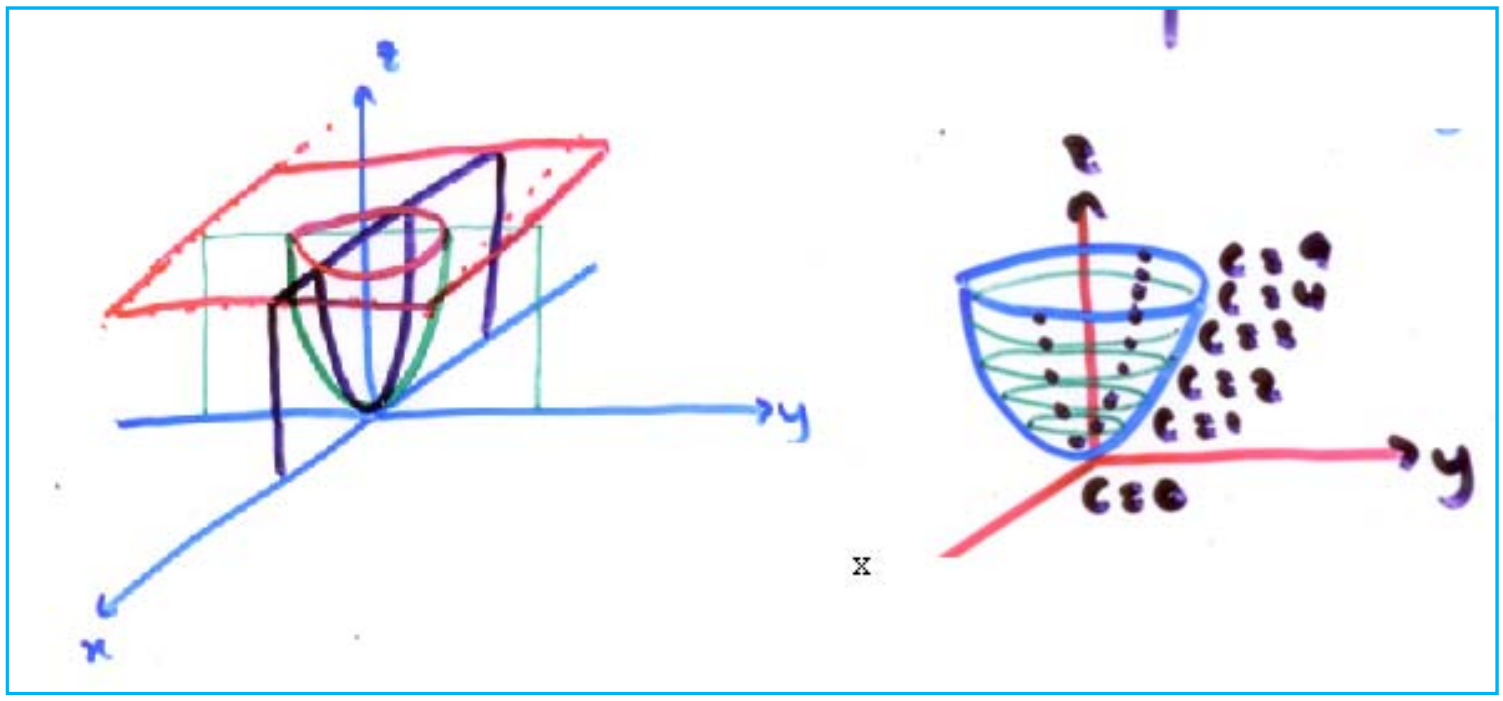
The traces in the coordinate planes:

- yz-plane, $x = 0$: the quadratic curve, $z = y^2$

- xz-plane, $y = 0$: the quadratic curve, $z = x^2$

- xy-plane, $z = 0$: a point (the origin)





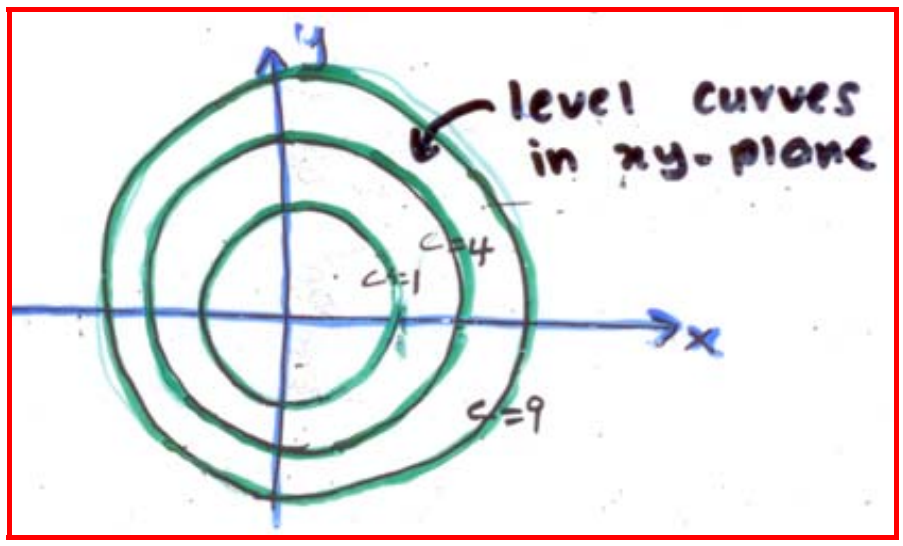
(ii) $z = \sqrt{x^2 + y^2}$, $c = 0, 1, 4, 9$.

$c = 0 \quad : \sqrt{x^2 + y^2} = 0$

$c = 1 \quad : \sqrt{x^2 + y^2} = 1$

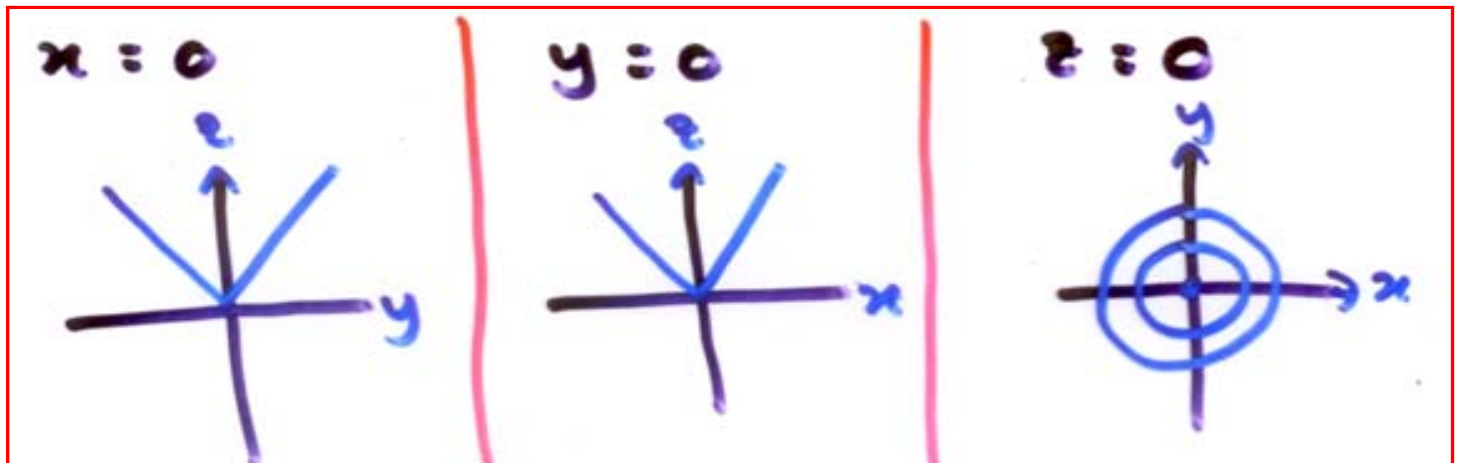
$c = 4 \quad : \sqrt{x^2 + y^2} = 4$

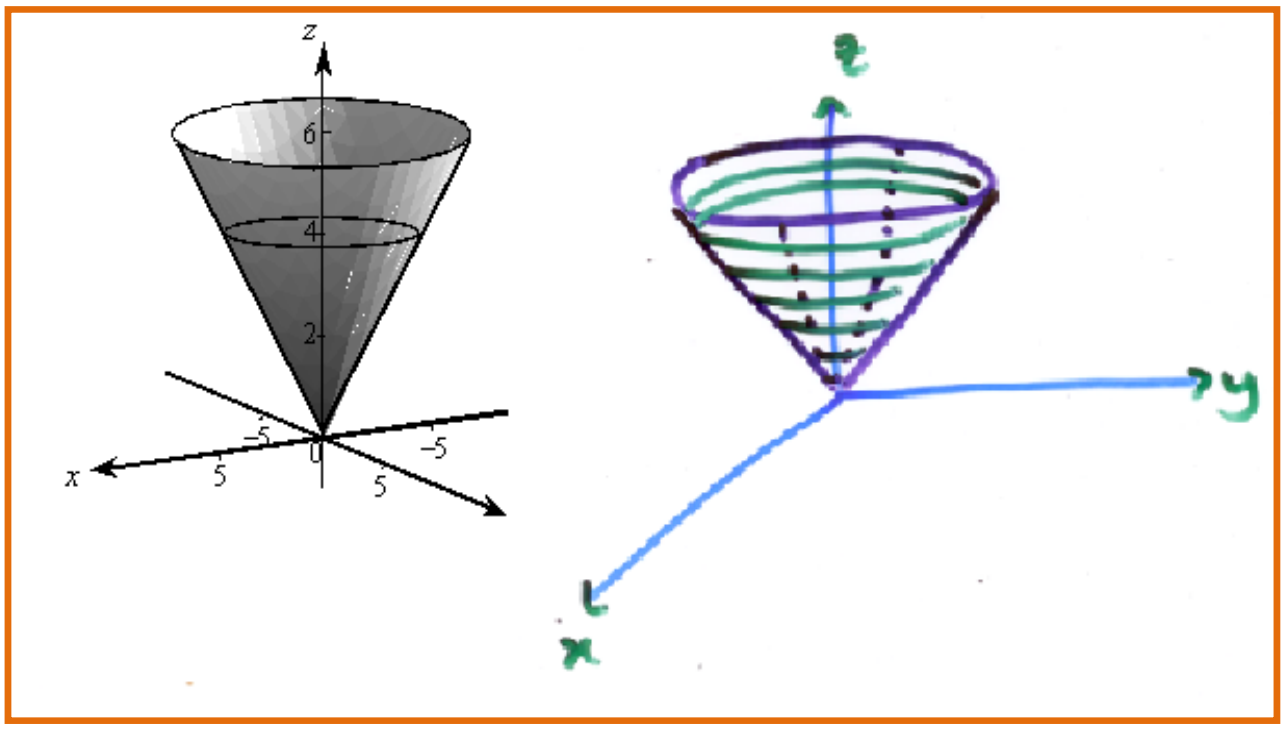
$c = 9 \quad : \sqrt{x^2 + y^2} = 9$



The traces in the coordinate planes:

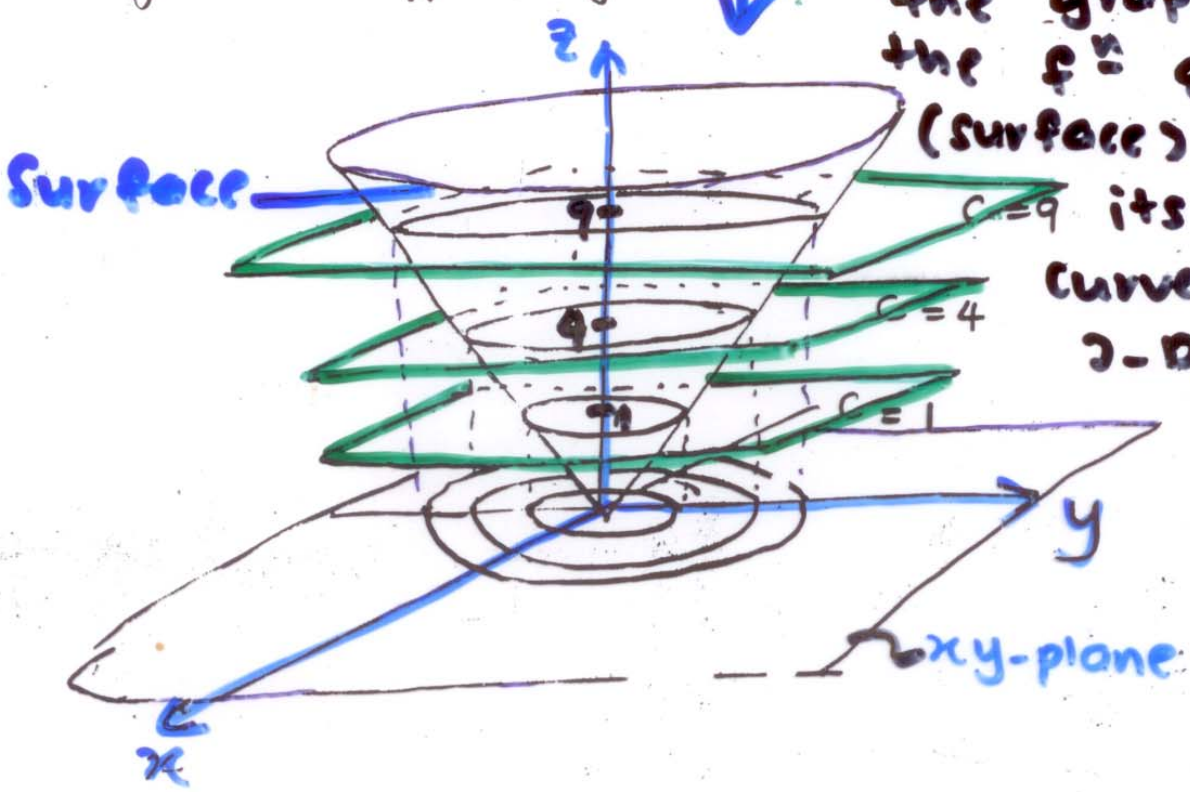
- yz-plane, $x = 0$: the straight line, $z = y$
- xz-plane, $y = 0$: the straight line, $z = x$
- xy-plane, $z = 0$: a point (the origin)
- parallel to xy-plane, $z = 4$: the circle $x^2 + y^2 = 4^2$


x = 0
z
y
y = 0
z
x
z = 0
y
x


z
6
4
2
x
-5
5
0
-5
5
z
y
x

eg. $z = \sqrt{x^2+y^2}$

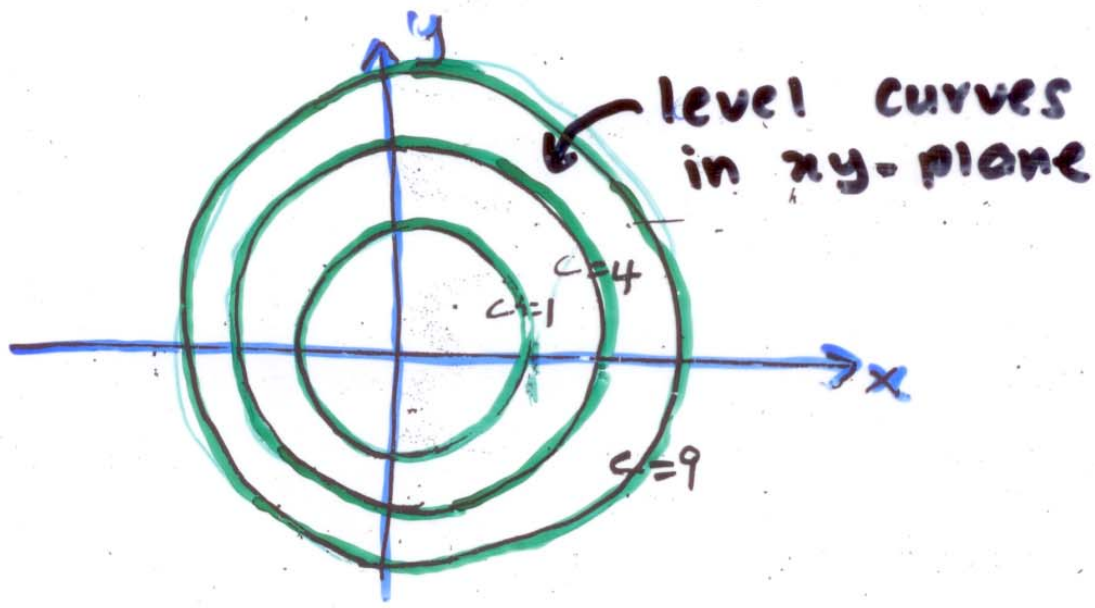
Relation between the graph of the f^n given (surface) and its level curves in 2-D



$c=1 \;:\; 1 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 1$

$c=4 \;:\; 4 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 16$

$c=9 \;:\; 9 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 81$



(ii) $z = 6 - x^2 - y$, $c = 0, 2, 4, 6$.

Sketching the level curves

- first, replace z with the value of c

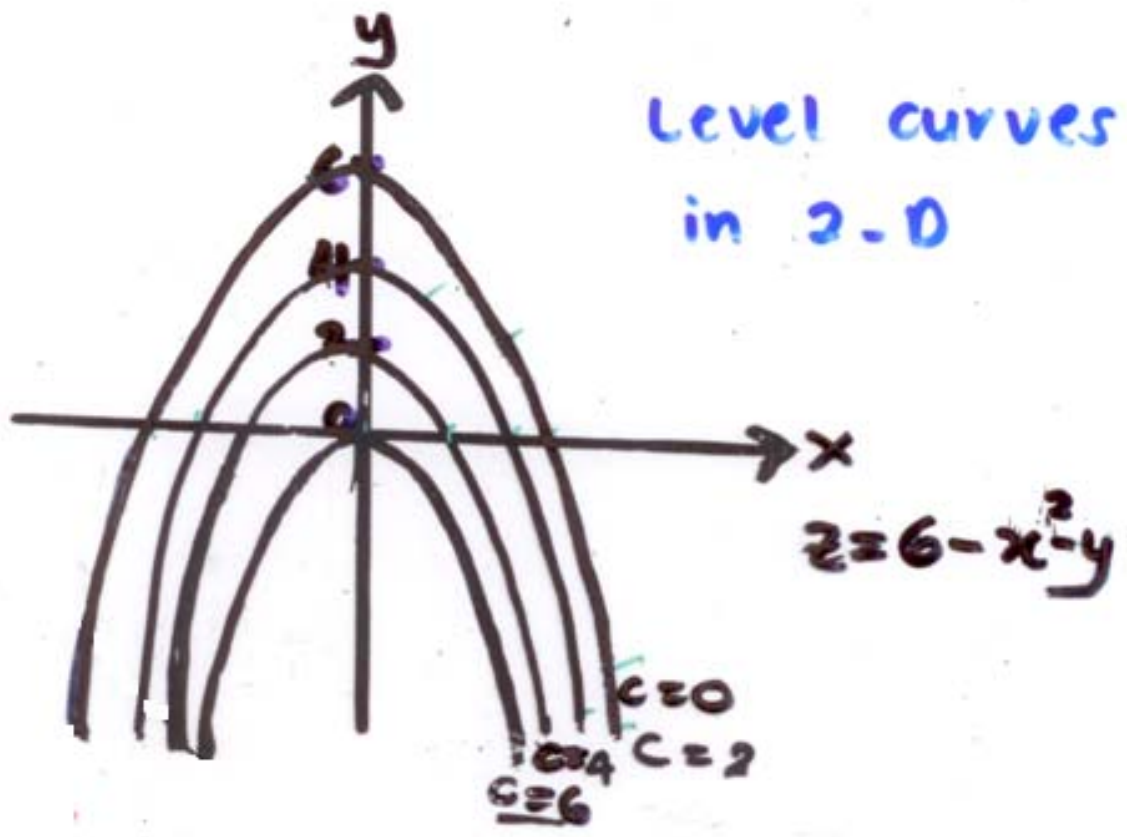
- second, plot the graph on the xy-plane

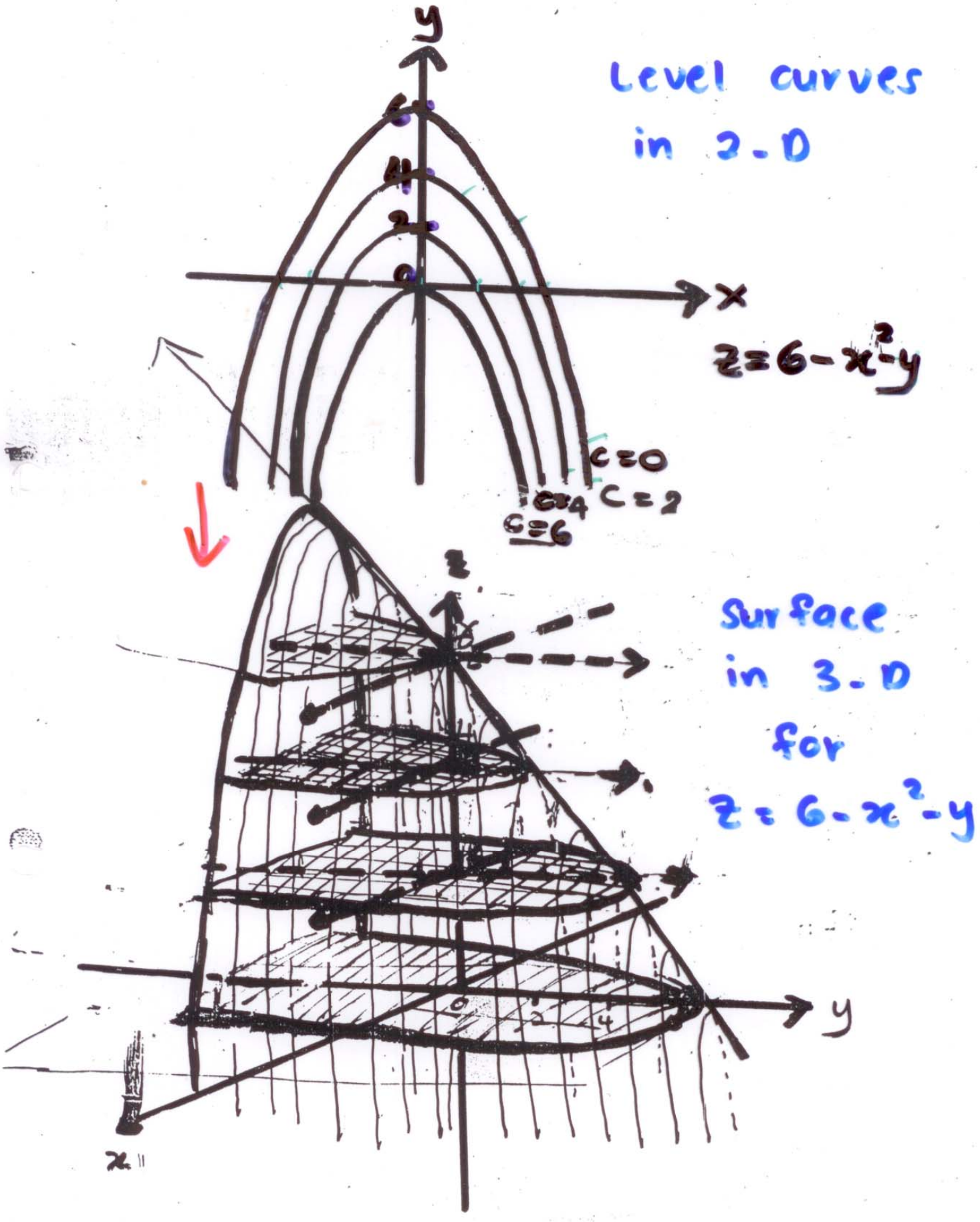
$c = 0 \quad : 6 - x^2 - y = 0 \Rightarrow y = -x^2 + 6$

$c = 2 \quad : 6 - x^2 - y = 2 \Rightarrow y = -x^2 + 4$

$c = 4 \quad : 6 - x^2 - y = 4 \Rightarrow y = -x^2 + 2$

$c = 6 \quad : 6 - x^2 - y = 6 \Rightarrow y = -x^2$




Level curves
in 2-D
y
x
6
4
2
0
$z = 6 - x^2 - y$
c=0
c=2
c=4
c=6
z
Surface
in 3-D
for
$z = 6 - x^2 - y$
0
2
4
y
x

1.1.5 Domain and Range of $z = f(x, y)$

$$\text{Domain} : \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}, \underline{???}\}$$

$\uparrow$

any constraint

??? $f(x, y)$ may consist:

***Sometimes we need to sketch the domain of the function given.**

Range – z-values that results when (x,y) varies over the domain

(i) z positive ?

(ii) z negative ?

(iii) z zero ?

(iv) z has maximum value ?

(v) z has minimum value ?

$$\text{Range}: \{z \mid z \in \mathbb{R},\ \underline{???}\}$$

put the limitation of z here!!

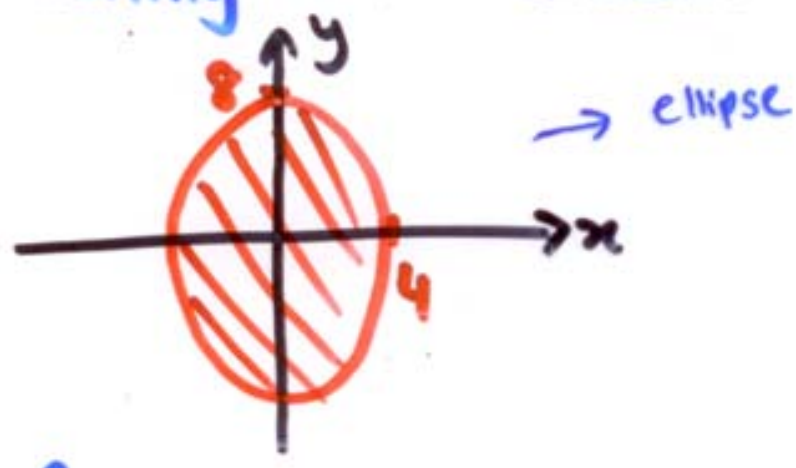
Example

Describe the domain and the range of

$$z = \sqrt{64 - 4x^2 - y^2}\ .$$

Solution

The sketching of Domain



Domain : $\{(x,y) / x \in \mathbb{R}, y \in \mathbb{R}, \ 64 - 4x^2 - y^2 \geq 0\}$

or

Domain : $\{(x,y) / x \in \mathbb{R}, y \in \mathbb{R}, \ \frac{x^2}{16} + \frac{y^2}{64} \leq 1\}$

Range : $\{ z / z \in \mathbb{R}, 0 \leq z \leq \sqrt{64} \}$

or

Range : $\{ z | z \in \mathbb{R}, 0 \leq z \leq 8 \}$

Example

Find the domain and range of $z = x^2\sqrt{y} - 1$.

Solution

Constraint : $y \geq 0$

Domain : $\{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R},\ y \geq 0\}$

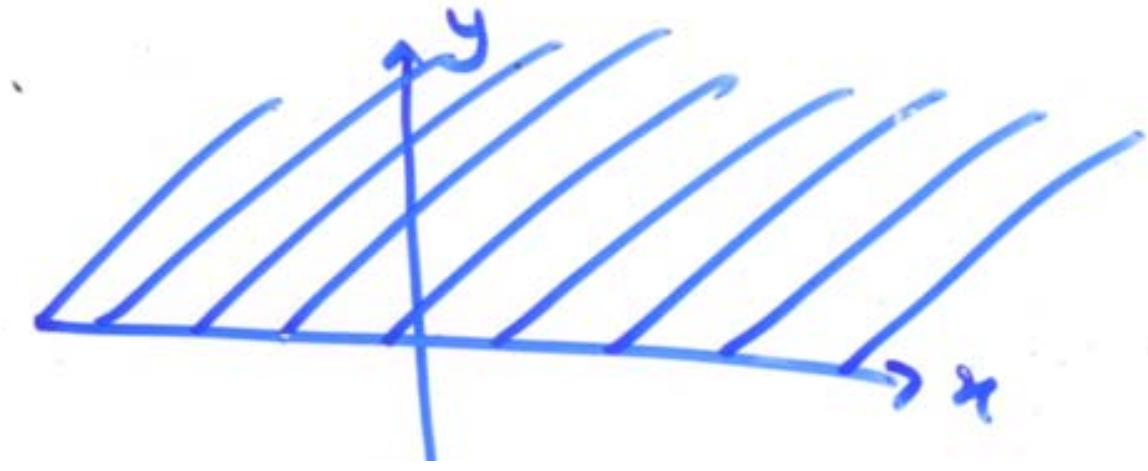
Range : ?

$x^2\sqrt{y} \geq 0$ (always +ve)

$\therefore \quad z \geq -1$

$\Rightarrow$ Range : $\{z \mid z \in \mathbb{R},\ z \geq -1\}$

Sketching of domain



Example

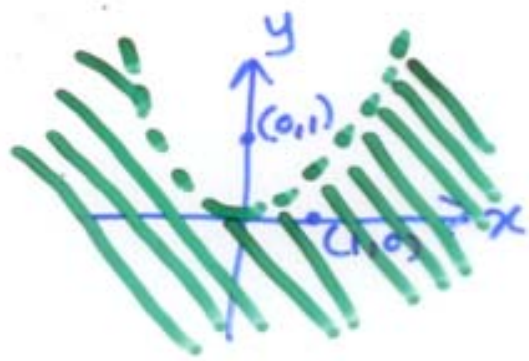
Find the domain and the range of $z = \ln(x^2 - y)$.

Solution

Constraint : $x^2 - y > 0 \Rightarrow y < x^2$

$\therefore$ Domain : $\{(x,y) \mid x \in R, y \in R, y < x^2\}$

Sketching of the Domain



For Range z can be zero
can be +ve
can be -ve

$\therefore$ Range : $\{z \mid z \in R, -\infty < z < +\infty\}$

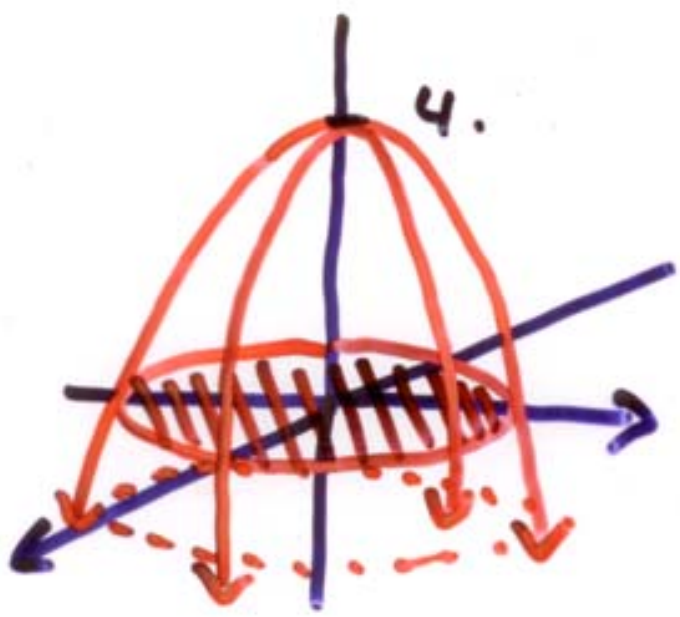
Example

Find the domain and the range of $z = 4 - x^2 - y^2$.

Solution

Domain $: \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

Range $: \{z \mid z \in \mathbb{R},\ z \le 4\}$



1.2 Functions of Three Variables

1.2.1 Domain and Range

Definition

A **function *f* of three variables** is a rule that assigns to each ordered triple (x, y, z) in some domain D in space a unique real number $w = f(x, y, z)$.

The range consists of the output values for w.

Example 1

Identify the domain and range for the following functions.

a). $w = \sqrt{x^2 + y^2 + z^2}$

$x^2 + y^2 + z^2 \geq 0$ for all points in space.

Domain : entire space

Domain : $\{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R},\ x^2 + y^2 + z^2 \geq 0\}$

Range : $[0, \infty)$

Range : $\{w \mid w \in \mathbb{R},\ w \geq 0\}$

b) $w = \sqrt{1 - \left(x^2 + y^2 + z^2\right)}$

We must have $1 - \left(x^2 + y^2 + z^2\right) \geq 0$ in order to have a real value for $f(x, y, z)$.

Rewriting the condition, we obtained

$$\boxed{x^2 + y^2 + z^2 \leq 1}$$

Thus the domain consists of all points on or within the sphere $x^2 + y^2 + z^2 = 1$, or

Domain : $\{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R},\ x^2 + y^2 + z^2 \leq 1\}$

Range : $[0, 1]$ or

Range : $\{w \mid w \in \mathbb{R},\ 0 \leq w \leq 1\}$

c) $w = \dfrac{1}{x^2 + y^2 + z^2}$

Domain : $\{(x, y, z) : (x, y, z) \neq (0, 0, 0)\}$
or

Domain : $\{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R},\ x^2 + y^2 + z^2 \neq 0\}$

Range : $(0, \infty)$ or

Range : $\{w \mid w \in \mathbb{R},\ w > 0\}$

d) $w = xy \ln z$

Domain : $\{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R},\ z > 0\}$

Range : $(-\infty, \infty)$ or

Range : $\{w \mid w \in \mathbb{R},\ -\infty \leq w \leq \infty\}$

1.2.2 Level Surfaces

The graphs of **functions of three variables** consist of points $(x, y, z, f(x, y, z))$ lying in **four-dimensional space**.

- Graphs cannot be sketch effectively in three-dimensional frame of reference.
- Can obtain insight of how function behaves by looking at its three-dimensional level surfaces.

The graph of the equation $f(x, y, z) = k$ will generally be a **surface in 3-space** which we call the **level surface** with constant k.

Remark

The term "level surface" is standard. It need **not** be level in the sense being horizontal; it is simply a surface on which all values of f are the same.

Example

Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$

(b) $f(x, y, z) = z^2 - x^2 - y^2$

Solution

(a) $f(x, y, z) = x^2 + y^2 + z^2$

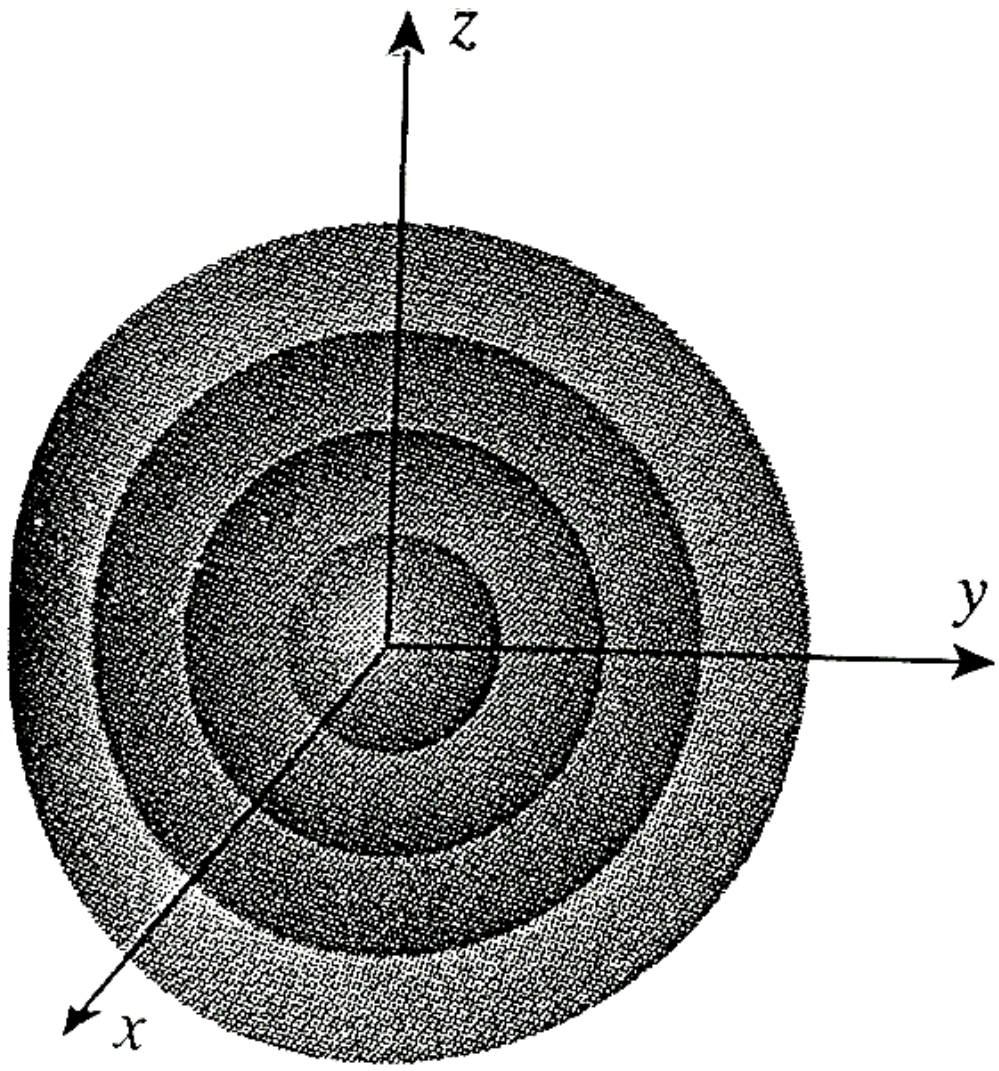
The level surfaces have equation of the form

$$x^2 + y^2 + z^2 = k$$

For $k > 0$, **the graph of this equation is a sphere of radius $\sqrt{k}$, centred at the origin**.

For $k = 0$, the graph is the **single point** $(0, 0, 0)$.

For $k < 0$, there is **no level surface**.



Level surfaces of

$$f(x, y, z) = x^2 + y^2 + z^2$$

b) $f(x, y, z) = z^2 - x^2 - y^2$

The level surface have equation of the form

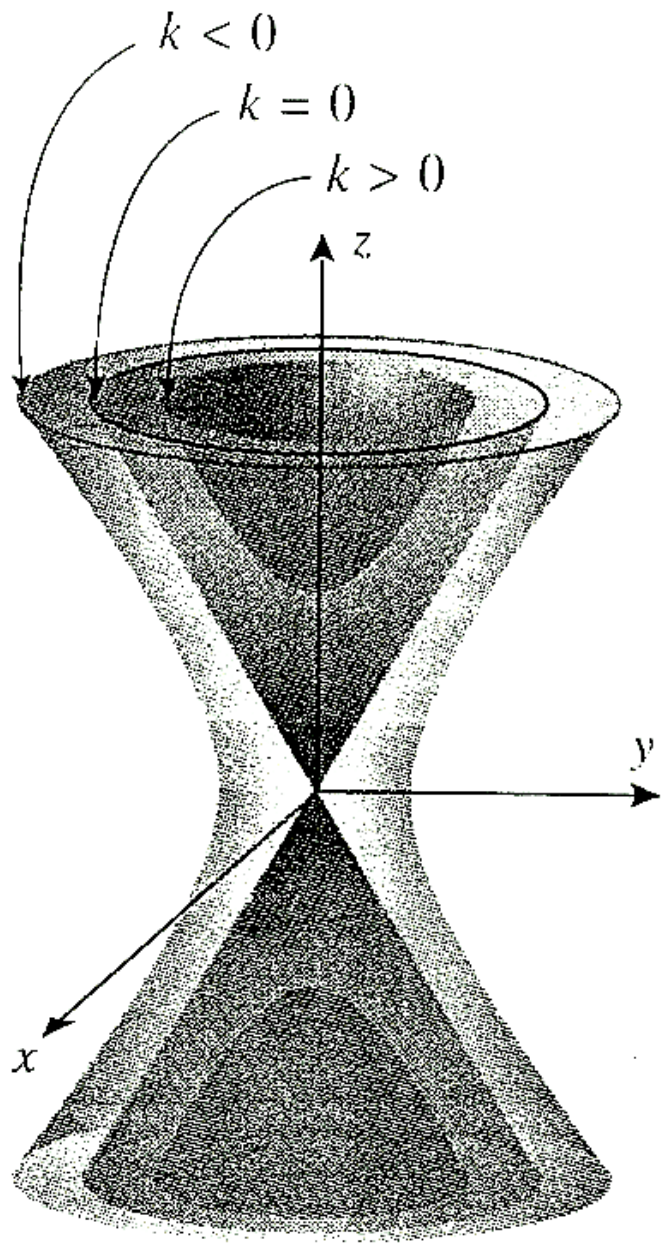
$$z^2 - x^2 - y^2 = k$$

For $k > 0$, the graph is a **hyperboloid of two sheets.**

For $k = 0$, the graph is a **cone.**

For $k < 0$, the graph is a **hyperboloid of one sheet.**

Level surfaces of



$$f(x, y, z) = z^2 - x^2 - y^2$$

Exersizes

Describe the level surfaces of

(i) $f(x, y, z) = x^2 + y^2$ **for** $C = 4,\ C = 9.$

(ii) $f(x, y, z) = 4x^2 + y^2 + 4z^2$ **for** $w = 1,\ w = 4.$