Chapter 6 Multiple Integral

6.1 Double Integrals

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6.1 Double Integrals

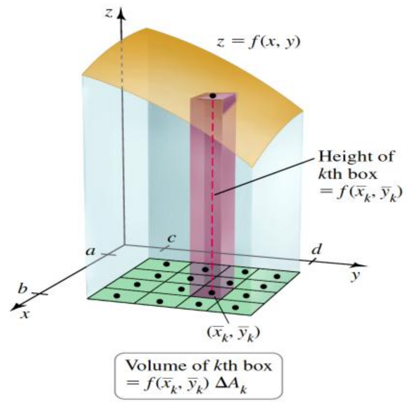
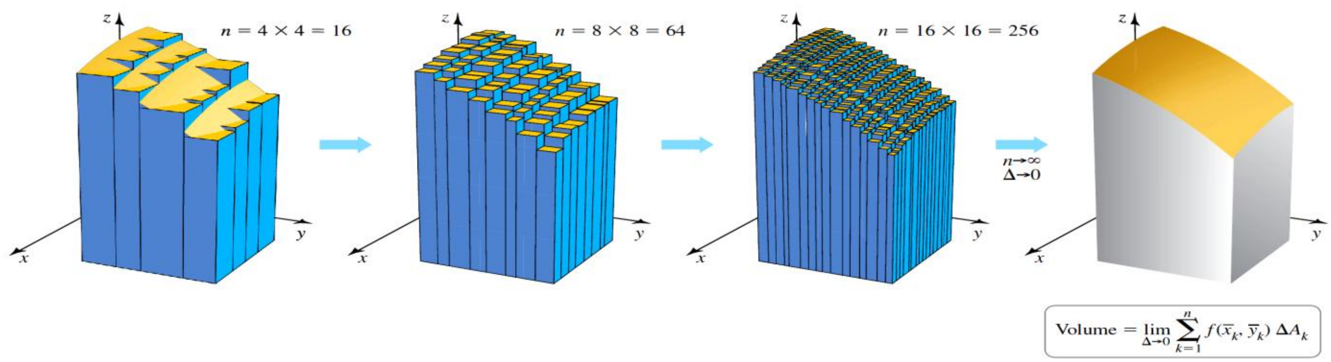


FIGURE 13.3



Definition

If f is a function of two variables that is defined on a region R in the xy-plane, then the **double integral of *f* over *R*** is given by

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\Delta A$$

provided this limit exists, in which case f is said to be **integrable** over R.

Note

- The double integral of the surface $z = f(x, y)$ is the volume between the region *R* and below the surface.
- The sum:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

is called the double **Riemann sum** and is used as an approximation to the value of the double integral.

The double integral inherits most of the properties of the single integral.

Properties of Double Integrals

1. constant multiple rule

$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$, *c* a constant

2. linear rule

$$\iint_R [f(x,y)+g(x,y)]dA$$

$$=\iint_R f(x,y)dA+\iint_R g(x,y)dA$$

3. subdivision rule

$$\iint_R f(x,y)dA=\iint_{R_1} f(x,y)dA+\iint_{R_1} f(x,y)dA$$

4. dominance rule, if $f(x,y)\geq g(x,y)$

$$\iint_R f(x,y)dA\geq\iint_R g(x,y)dA$$

6.2 Iterated Integrals

Evaluating Double Integrals

- It is impractical to obtain the value of double integral from the definition. We evaluate the integrals by calculating two successive ***single*** integrals.

We use the notation $\int_c^d f(x,y)\,dy$ to mean that *x* is held fixed and $f(x, y)$ is integrated with respect to *y* from $y = c$ to $y = d$. This is called **partial integration** with respect to *y*.

$$A(x) = \int_c^d f(x,y)\,dy$$

Now we integrate the function *A* with respect to *x* from $x = a$ to $x = b$, we get:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

This successive integration process is called ***iterated integration***.

$$\iint f(x,y)dxdy = \int\left[\int f(x,y)dx\right]dy$$

$$\iint f(x,y)dydx = \int\left[\int f(x,y)dy\right]dx$$

- These iterated integrals mean that we first integrate with respect to one variable (while holding the other fixed) and then integrating with respect to the other variable while holding the first one fixed.

- It is traditional to omit the brackets and write the iterated integral simply as

$$\iint f(x,y)dxdy$$

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Question

In questions a) - c), evaluate the iterated integrals.

(a) $\int_{-1}^{0}\int_{0}^{1} \; x - y \;\;^{2} \, dx \, dy$

(b) $\int_{0}^{\pi/2}\int_{0}^{a} \; r \sin \;\theta\; + \cos\; \theta \;\; dr \, d\theta$

(c) $\int_{0}^{3}\int_{0}^{1} x\sqrt{x^2 + y}\, dx \, dy$

Theorem : Fubini's Theorem

If $f(x,y)$ is continuous over the ***rectangle***
$R: a \le x \le b, c \le y \le c$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example

Evaluate the integrals.

(a) $\displaystyle\int_0^3 \int_1^2 (1 + 8xy)\,dy\,dx$ (b) $\displaystyle\int_1^2 \int_0^3 (1 + 8xy)\,dx\,dy$

Compare (a) and (b). What can you say about the integration?

Solution

(a) $\displaystyle\int_0^3 \int_1^2 (1+8xy)\,dy\,dx = \int_0^3 \left[\int_1^2 (1+8xy)\,dy \right] dx$

$$= \int_0^3 \left[y + 4xy^2 \right]_1^2 dx$$

$$= \int_0^3 1 + 12x dx$$

$$= x + 6x^2 \Big|_0^3 = 57$$

(b) $\int_1^2 \int_0^3 (1+8xy) dx dy = \int_1^2 \left[\int_0^3 (1+8xy) dx \right] dy$

$$= \int_1^2 \left[x + 4x^2 y \right]_0^3 dy$$

$$= \int_1^2 3 + 36y dy$$

$$= 3y + 18y^2 \Big|_1^2 = 57$$

Nonrectangular Regions

We limit our study of double integrals to two basic types of regions: Type I and Type II.

Definition

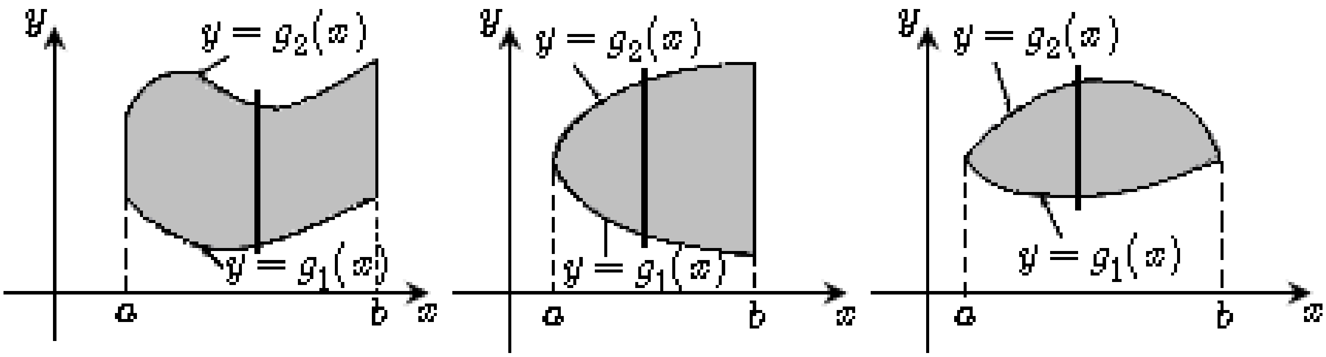
(a) A plane region *R* is said to be of **Type I** if it lies between the graphs of two continuous functions of *x*.

$$R = \ (x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)$$

(b) A plane region *R* is said to be of **Type II** if it lies between the graphs of two continuous functions of *y*.

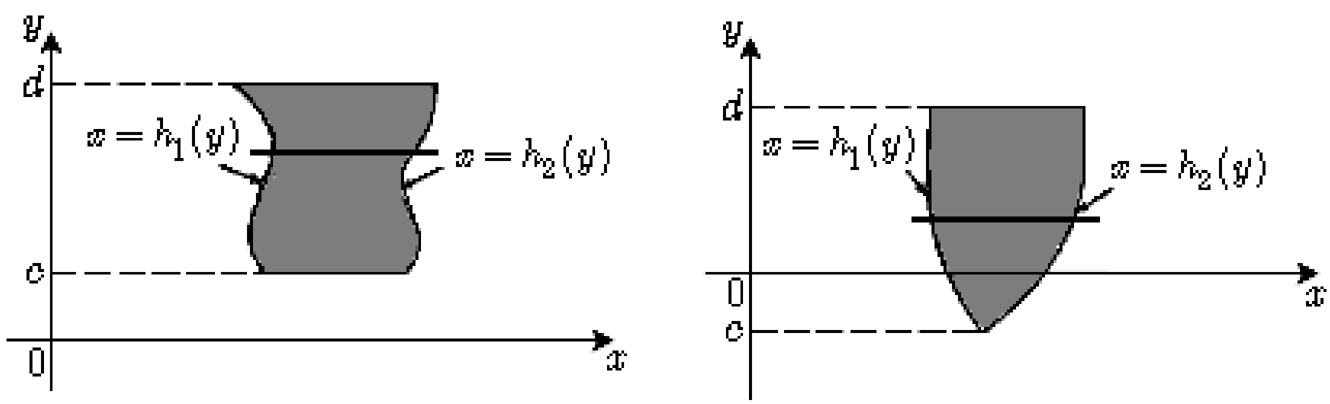
$$R = \ (x,y) : h_1(y) \le x \le h_2(y), c \le y \le d$$

Type I Region - integrating first with respect to *y*



Type I (Vertical Strip): *x* fixed between *a* and *b*, *y* varies from $g_1(x)$ to $g_2(x)$.

Type II Region - integrating first with respect to *x*



Type II (Horizontal Strip): *y* fixed between *c* and *d*, *x* varies from $h_1(y)$ to $h_2(y)$.

Theorem

(a) If R is a **Type I** region, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(b) If R is a **Type II** region, then

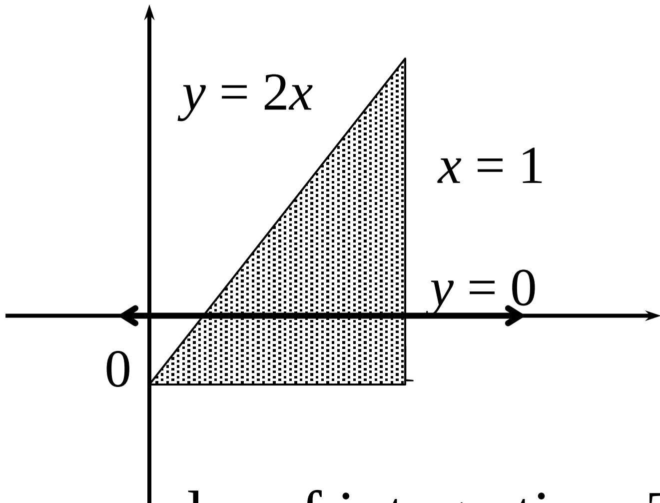
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example

Evaluate $\iint_R (x+y)\,dA$ over the region R enclosed by the lines $y=0$, $y=2x$ and $x=1$.

Solution

- Sketch the region: set up the limits of integration



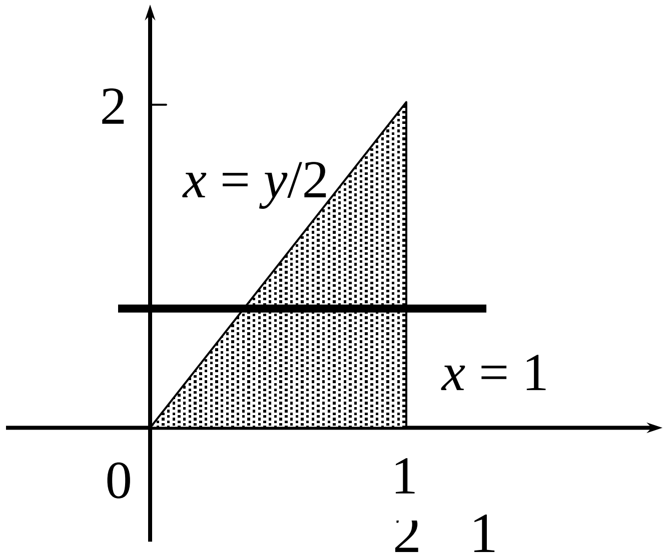
Choose order of integration: Type I, fixed x

$$\iint_R (x+y)\,dA = \int_0^1 \int_0^{2x} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=2x} dx = \int_0^1 4x^2\,dx$$

$$= \frac{4}{3}x^3 \Big|_{x=0}^{x=1} = \frac{4}{3}$$

Alternatively, reversing the order of integration: Type II, fixed y



$$\iint_R (x+y)\,dA = \int_0^2 \int_{y/2}^1 (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=y/2}^{x=1} dy$$

$$= \int_0^1 \left[\frac{1}{2} + y - \frac{5y^2}{8}\right] dy$$

$$= \frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24}\bigg|_{y=0}^{y=2} = \frac{4}{3}$$

Double Integral as Area and Volume

Definition

(a) The **area** of the region *R* in the *xy*-plane is given by

$$A = \iint\limits_R dA$$

(b) If *f* is continuous and $f(x,y) \geq 0$ on the region *R*, the **volume** of the solid under the surface $z = f(x,y)$ above the region *R* is given by

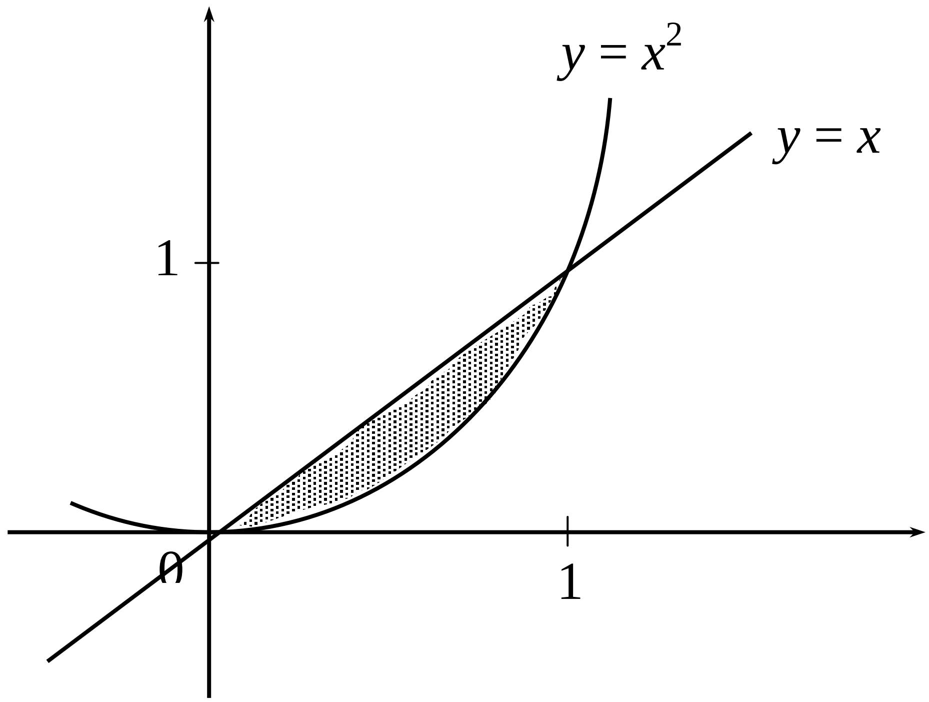
$$V = \iint\limits_R f(x,y)\,dA$$

Example

Find the area of the region bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

Sketch the region:



Order of integration: Type I, fixed *x*

Area = $\int_0^1 \int_{x^2}^{x} dy dx = \int_0^1 [y]_{x^2}^{x} \, dx$

$$= \int_0^1 x - x^2 \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6} \text{ unit}^2$$

Question 1

In questions 1(a) - 1(c), evaluate $\iint_R f \;\; x,y \;\; dA.$

(a) $f \;\; x,y \;\; = 12xy^2$
$; R = \quad x,y \;\; : 1 \le x \le 2, -1 \le y \le 2 \;\; .$

(b) $f \;\; x,y \;\; = 4 - y^2$ where *R* is the closed rectangular region with vertices (0,0), (3,0), (2,4) and (-1,4).

(c) $f \;\; x,y \;\; = 3x + 2y$
$; R = \quad x,y \;\; : 0 \le x \le 1, x^3 \le y \le -x^2 \;\; .$

Question 2

In questions 2(a) - 2(b), sketch the closed region bounded by the given curves, and find the area of the region using a double integral.

(a) $y = \sqrt{x}, y = -x, x = 1, x = 4.$

(b) $x = y^2, y - x = 2, y = -2, y = 3.$

Question 3

In questions 3(a) - 3(b), sketch the solid in the first octant bounded by the given surfaces, and find its volume by using a double integral.

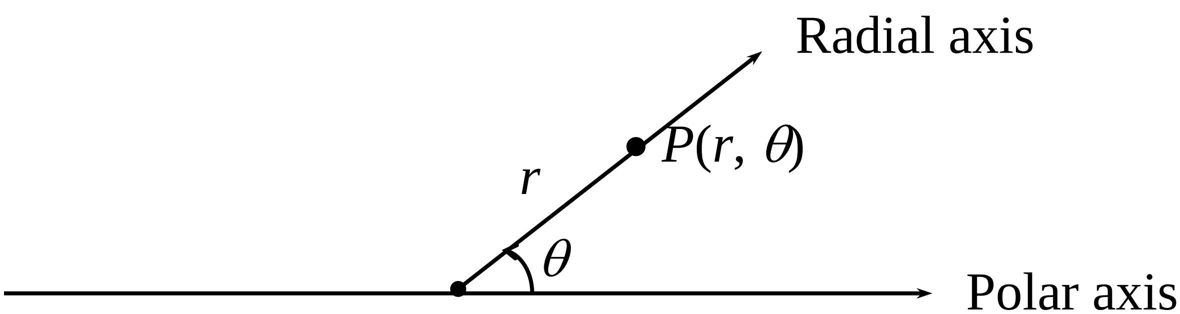
(a) $2x + y + z = 4, x = 0, y = 0, z = 0.$

(b) $z = 4 - x^2, x + y = 2, x = 0, y = 0, z = 0.$

6.3 Double Integral in Polar Form

Polar Coordinates System

A polar coordinate system consist of a fixed point O called the origin or **pole** and a line segment starting from the pole called the **polar axis**.

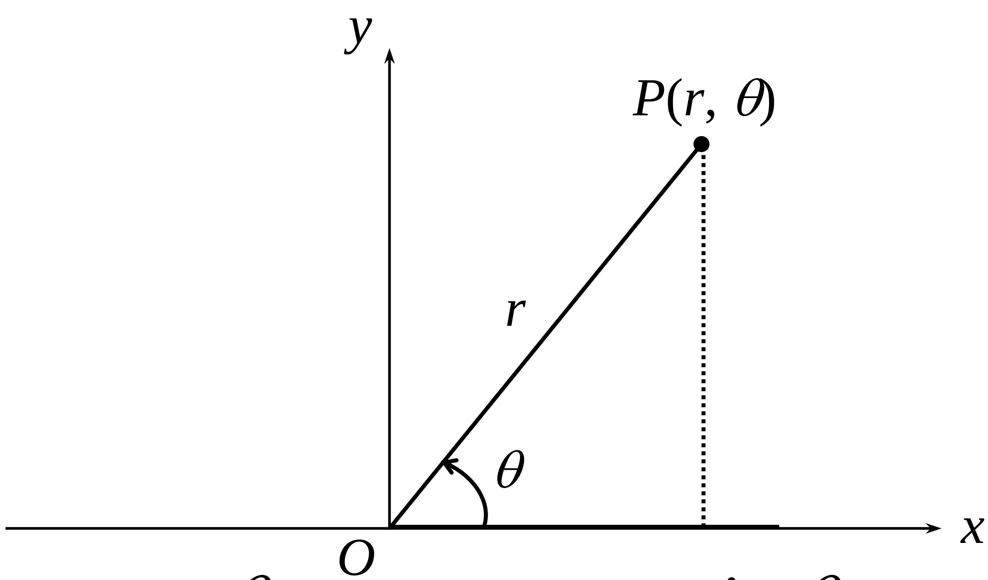


r – radial coordinate
θ – polar angle

Definition

Polar coordinates of a point P is written as (r,θ) where r is the distance of P from the pole and θ is the angle measured from the polar axis to the line OP (radial axis).

Relationship between Polar and Cartesian Coordinates



$$x = r\cos\theta \qquad y = r\sin\theta,$$

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$$

Note

(i) Polar coordinate of a point is not unique.

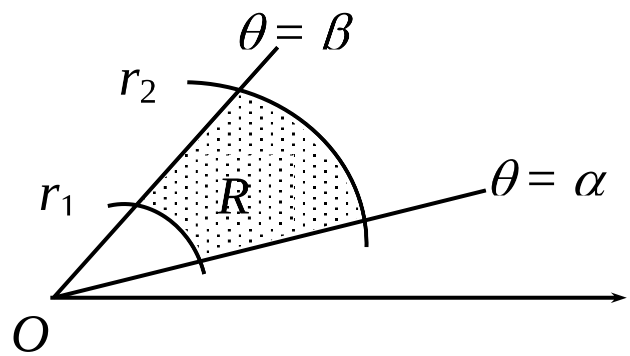
(ii) θ is positive in an anticlockwise direction, and negative if it is taken clockwise.

(iii) A point $(-r, \theta)$ is in the opposite direction of point (r, θ).

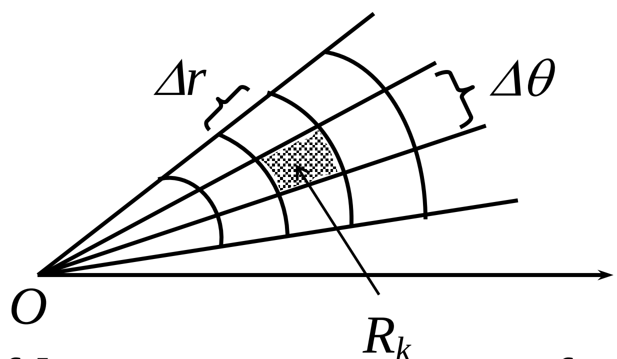
Integrals in Polar Coordinates

If R is a circular region (involves $x^2 + y^2$), it is easily described using polar coordinates.

- Divide the region into polar rectangles.



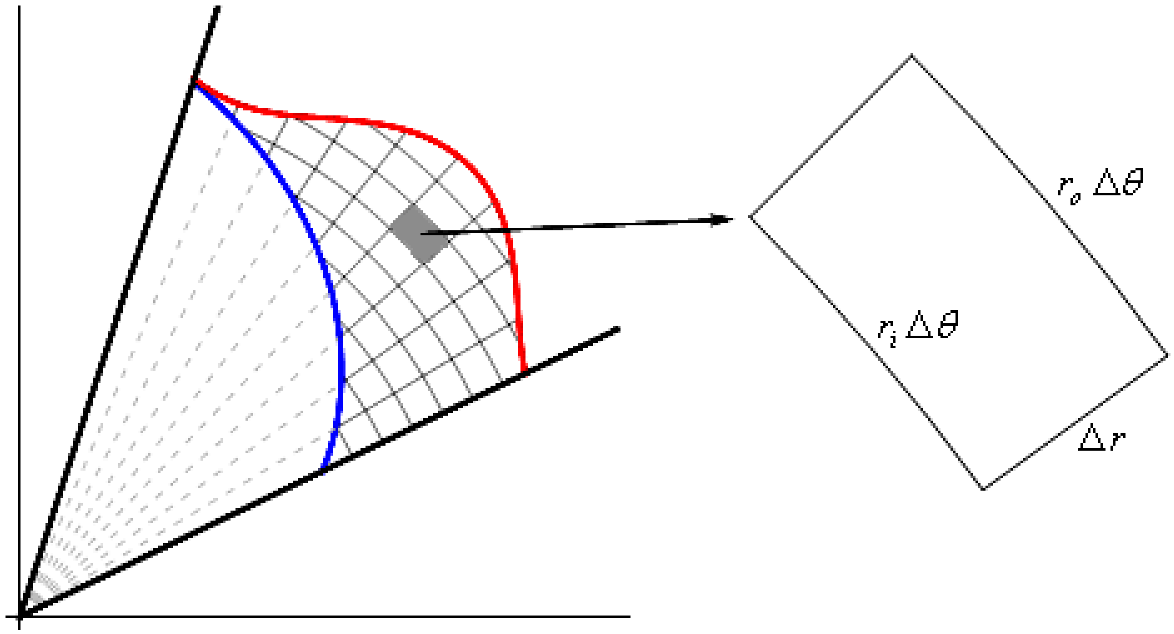
- Find the area of typical polar rectangle:



ΔA_k = area of large sector – area of small sector

$$= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = r_k \Delta r_k \Delta\theta_k$$

Alternatively:



If the mesh is small enough, we can assume that,

$$r_0 \approx r_1 = r$$

and with this assumption we can also assume that our polar slab is close enough to a rectangle,

$$\Delta A \approx r \Delta\theta \Delta r$$

Thinking of volume, we make the equation $z = f(r\cos\theta, r\sin\theta)$, thus the Riemann sum can be written as:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^*, \theta_j^*)\, r^* \Delta r \Delta\theta$$

Taking the limit we have the actual volume,

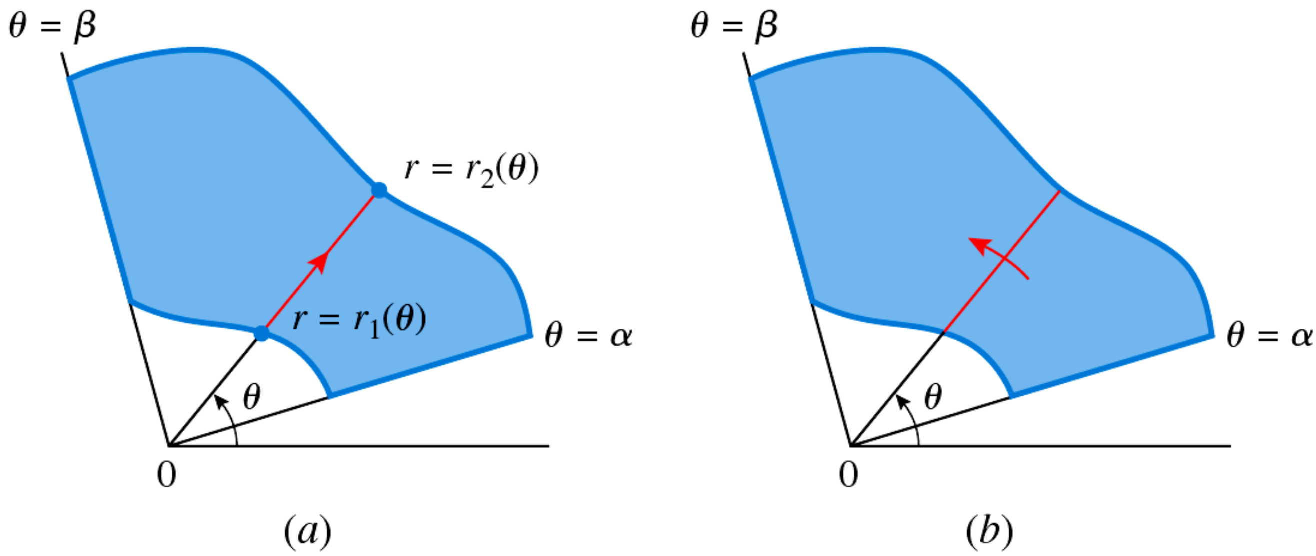
$$\int\int_R f(x,y)\,dA = \int\int_R f(r,\theta)\,r\,drd\theta$$

A version of Fubini's Theorem now says that the integral can be evaluated by iteration with respect to r and θ.

Theorem

Let R be a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$. If $f(r,\theta)$ is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\,r\,drd\theta$$



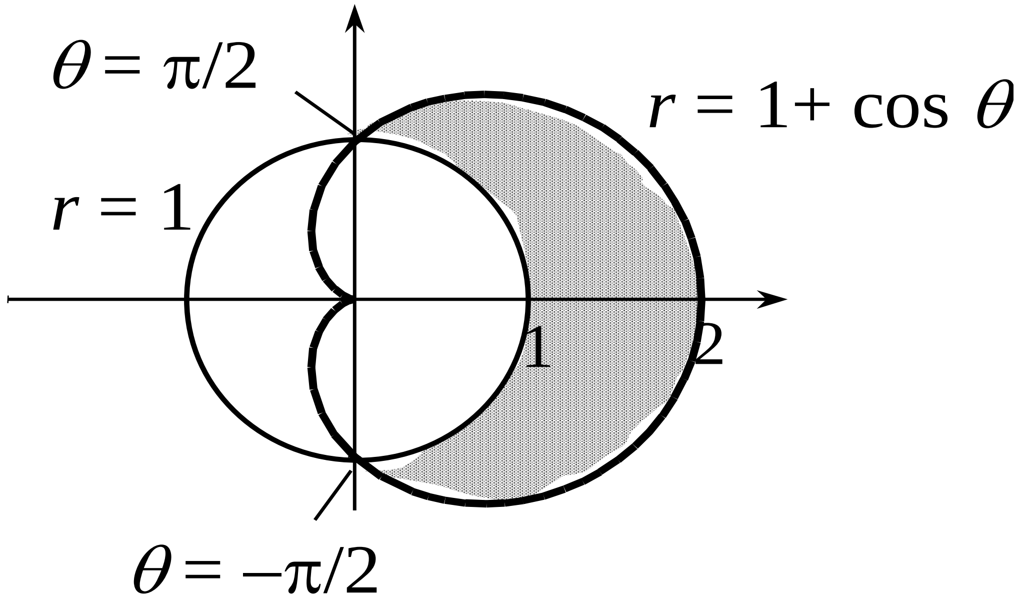
Finding limits of Integration

Example

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardiod $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

Step 1: Sketch R



Step 2: the r-limits of integration

A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

Step 3: the θ-limits of integration

The rays from the origin that intersect R run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr d\theta$$

Note

We may, of course, integrate first with respect to θ and then with respect to r if this is more convenient.

Changing Cartesian Integrals into Polar Integrals

The procedure for changing Cartesian integral $\iint_R f(x,y)\,dA$ into a polar integral has two steps.

Step 1: Substitute $x = r\cos\theta$, $y = r\sin\theta$ and replace $dxdy$ by $r\,drd\theta$ in the Cartesian integral.

Step 2: Supply polar limits of integration for the boundary of *R*.

The Cartesian integral then becomes

$$\iint_R f(r,\theta)\,dA = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example

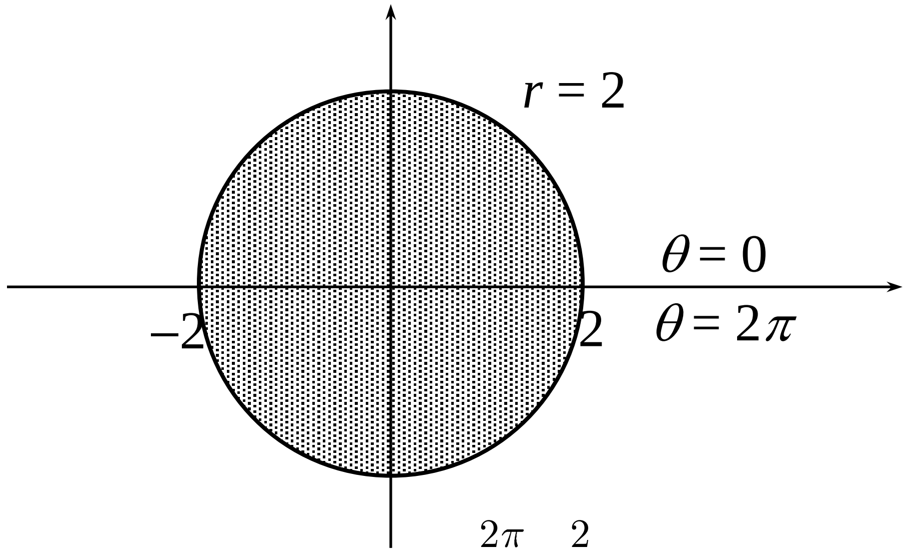
Evaluate $\iint_R (x^2+y^2+1)\,dA$ where *R* is the region inside the circle $x^2+y^2=4$.

Solution

We evaluate the integral in polar form.

KNOW: $x^2 + y^2 = r^2$

Region *R*: $x^2 + y^2 = 4 \Rightarrow r^2 = 4$ or $r = 2$



$$\iint_R (x^2 + y^2 + 1)\, dA = \int_0^{2\pi} \int_0^{2} (r^2 + 1)\, r \, dr d\theta$$

Question 1

In questions 1(a) and 1(b), evaluate the double integral.

(a) $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\sin\theta} r\cos\theta\, dr d\theta$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2}\int_{0}^{a\sin\theta} r^2\, dr d\theta$

Question 2

Sketch the closed region bounded by the polar equations, and find its area by using a double integral in polar coordinates.

(a) $r = 2, r = 4\sin\theta, \theta = \dfrac{\pi}{3}, \theta = \dfrac{2\pi}{3}$.

(b) The region inside the cardiod $r = 2\ \ 1+\sin\theta$ and outside the circle $r = 3$.

Question 3

Evaluate the integrals by changing to polar coordinates.

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} e^{-\ x^2+y^2} \ dy dx$$

Question 4

Find the volume of the solid bounded by $z = 9 - x^2 - y^2$ and $z = 5$.