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SSH 1033 MATHEMATICAL METHODS 2 **TUTORIAL 7**

1. Given $z = f(x, y) = x^2y - 3y$.
   (a) Find the expressions for Δz and dz.
   (b) Determine Δz and dz if $x = 4$, $y = 3$, $\Delta x = -0.001$, $\Delta y = 0.02$.
   (c) How might you determine $f(5.12, 6.85)$ without direct computation?

2. Given $z = x^3 - xy + 3y^2$.
   (a) Compute Δz and dz where $x = 5$, $y = 4$, $\Delta x = -0.2$, $\Delta y = 0.1$.
   (b) Find Δz and dz if $x = 5$, $y = 4$, $\Delta x = -2$, $\Delta y = 1$.

3. Use partial derivative to find the change in the value of $f(x, y) = x^2y - xy^2$ as (x, y) changes from $(1, 1)$ to $(1.02, 0.98)$.

4. Estimate the change in the value of $f(x, y, z) = 2xy^2z^3$ when (x, y, z) changes from $P(1, -1, 2)$ to $Q(0.99, -1.02, 2.02)$.

5. Use differentials to find the approximate value of the following expressions.
   (a) $\sqrt{(5.02)^2 + (11.97)^2}$. (b) $\sqrt{(3.02)^2 + (1.99)^2 + (5.97)^2}$.

6. A box has square ends, 11.98 cm on each side, and has a length of 30.03 cm. Find its approximate volume, using differentials.

7. The period T of a simple pendulum is given by $T = 2\pi\sqrt{\ell/g}$, where ℓ is the length and g is the gravitional constant. If ℓ is measured to be 20 cm with an error of 0.2, if g is 980 with an error of 7, and π is computed as 3.14 with an error of 0.002, use differentials to find an approximate value of T.

8. The legs of a right triangle are measured and found to be 6.0 and 8.0 cm, with possible error of 0.1 cm. Find approximately the maximum possible value of error in computing the hypotenuse. What is the maximum approximate percentage error?

9. Find in degrees the maximum possible approximate error in the computed value of the smaller acute angle in the triangle of Question 6.

10. By measurement, a triangle is found to have two sides of length 50 cm and 70 cm; the angle between them is 30°. If there are possible errors of $\frac{1}{2}$% in the measurements of the sides and $\frac{1}{2}$ degree in that of the angle, find the maximum approximate percentage error in the measurement of the area?

11. Let $I = V/R$, find
    (a) the error in calculating I if the error in computing V is 1 volt and R is 0.5 ohm at $V = 250$ volt and $R = 50$ ohm.
    (b) the maximum percentage error in evaluating I, if the percentage error in estimating V and R are 2% and 1% respectively.

12. By using the partial derivatives, estimate the maximum percentage error in evaluating $T = 2\pi\sqrt{\ell/g}$, if the percentage error in estimating ℓ and g are 0.5% and 0.1% respectively.

13. The dimensions of a closed rectangular box are measured as 3 m, 4 m and 5 m with a possible error of 100/192 cm in each case. Use partial derivatives to approximate the maximum error in calculating the value of

(a) the surface area of the box.

(b) the volume of the box.

14. The flow rate of gas through a pipe is given by $V = cp^{1/2}T^{-5/6}$, where c is a constant, p is diameter of the pipe and T the absolute temperature of the gas. The value of p is measured with a maximum percentage error of 1.6% while the maximum percentage error in T is 0.36%. Find the maximum percentage error in calculating V.

15. A box with height h has a square base with length x. The error in measuring the side of the base is 1% whereas that for the height is 2%. Approximate the maximum percentage error in calculating the volume.

16. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Show that for $i = 1, 2, 3$, $\dfrac{\partial R}{\partial R_i} = \left(\dfrac{R}{R_i}\right)^2$. Hence, find the maximum percentage error in calculating R if

(a) the percentage error in measuring the resistance R_1, R_2 and R_3 is 2% respectively.

(b) R_1, R_2 and R_3 are given as 100, 200 and 400 ohm with the maximum percentage error of 2%, 1%, 2%, for each measurements.

17. The radius of a right-circular cylinder is measured with an error of at most 2%, and the height is measured with an error at most 4%. Approximate the maximum possible percentage error in the volume V calculated from these measurements.

18. Derive an expression to estimate the possible error in S calculated from

$$S = \frac{1}{2}\, b c \sin A,$$

when there are errors δb, δc and δA in b, c and A. Use your expression to calculate the error when $b = 5$, $c = 2$ and $A = 30°$, where b and c may each be in error by 0.05 and A by $10'$.

19. Derive an expression to estimate the possible error in a calculated from

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

when there are errors δb, δc and δA in b, c and A. Use your expression to calculate the error when $b = 4.10$, $c = 3.95$ and $A = 62°$.

20. Find the equation of tangent plane and the equations of the normal line to the given surface at the given point.

(a) $z = x^2 + 2y^2$; $(2, -1, 6)$. (b) $z = 3x^2 - y^2 - 2$; $(-1, 2, -3)$.

(c) $z = e^x \sin y$; $(1, \pi/2, e)$. (d) $z = e^{2x} \cos 3y$; $(1, \pi/3, -e^2)$.

(e) $z = \ln \sqrt{x^2 + y^2}$; $(-3, 4, \ln 5)$. (f) $z = \ln(xy)$; $(e, e, 2)$.

21. Find the point or points (if any) on the surface

$$z = x^2 - 2y^2 + 3y - 6$$

where the tangent plane is parallel to the plane $2x + 3y + z = 5$.

22. Find the point or points (if any) on the surface

$$z = x^2 + 2xy - y^2 + 3x - 2y - 4$$

where the tangent plane is parallel to the xy–plane.

23. Find the point or points (if any) on the surface

$$x^2 + 2xy - y^2 + 3z^2 - 2x + 2y - 6z - 2 = 0$$

where the tangent plane is parallel to the yz–plane.

24. Find the critical points of the following functions and detemine their nature.

(a) $f(x, y) = 2x^2 - 2xy + 3y^2 - 4x - 8y$. (b) $f(x, y) = x^3 + 3xy^2 - 3x$.

(c) $f(x, y) = 2x^4 + y^2 - 12xy$. (d) $f(x, y) = e^x \cos y$.

(e) $f(x, y) = (x - 1)(y - 1)(x + y - 1)$. (f) $f(x, y) = e^x \left(x^2 + y^2\right)$.

(g) $f(x, y) = e^y(x^2 - xy)$. (h) $f(x, y) = 4xy - x^4 - y^4$.

25. Show that $f(x, y) = x^3 + y^3 - 2\left(x^2 + y^2\right) + 3xy$ has critical points at $(0, 0)$ and $(\frac{1}{3}, \frac{1}{3})$, and investigate their nature.

26. Locate and identify the stationary points of

(a) $f(x, y) = 2x^3 - 9x^2y + 12xy + 60y$. (b) $f(x, y) = 4xy - 2x^2 - y^4 + 3$.

27. Locate and identify the stationary points of

$$f(x, y) = x^2 - 12y^2 + 4y^3 + 3y^4,$$

showing that one is a saddle point. Sketch the graphs of $z = f(x, 0)$ and $z = f(0, y)$.

28. Find the (infinitely many) critical points of the surface

$$z = \sin\left(\frac{1}{2}\pi x\right) \sin\left(\frac{1}{2}\pi y\right).$$

Show that these points are saddle points when both x and y have even integer values.

29. Find the dimensions of the rectangular box, open at the top,

(a) is to have a volume of $32m^3$ so that the surface area is a minimum.

(b) which has maximum volume if the surface area is 12.

30. Use Lagrange multipliers to find the dimensions of the rectangular box with the largest volume if the total surface area is given as 64 cm^2.

31. A closed rectagular box is to be build with the volume equal to $36m^3$. The material for the bottom of the box cost RM 1 per m^2 while the top and sides cost RM 8 per m^2. Find the dimensions of the box with minimum cost.

32. Find the absolute maximum and minimum values of

$$f(x,\, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region $\mathcal{R}$ with vertices (0, 0), (3, 0), and (0, 5).

33. Find the critical points of the function

$$f(x, y) = e^{x+y}\left(x^2 - xy + y^2\right).$$

(a) Show that it has a minimum and a saddle point.

(b) Which is the point in the region $0 \le x \le 1$, $0 \le y \le 1$, where $f(x, y)$ reaches its maximum value? Determine the value.

34. If $z = \sin x \sin y \sin(x + y)$, show that

$$\frac{\partial z}{\partial x} = \sin y \sin(2x + y), \quad \frac{\partial z}{\partial y} = \sin x \sin(2y + x).$$

(a) Verify that the function

$$z = \sin x \sin y \sin(x + y)$$

has a critical value at $(x, y) = (\frac{1}{3}\pi, \frac{1}{3}\pi)$, and that is a maximum. (Note that you are **not** asked to examine other extrema).

(b) Show that z has a maximum value $\frac{1}{8}3\sqrt{3}$ at a point within the square whose sides at the lines $x = 0$, $x = \frac{1}{2}\pi$, $y = 0$, $y = \frac{1}{2}\pi$.

35. Use Lagrange multipliers to find the shortest distance from the point $(2, 1, -1)$ to the plane $x + y - z = 1$.

36. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$.

37. Use Lagrange multipliers to find three positive numbers whose sum is 100 and whose product is a maximum.

38. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

39. Use Lagrange multipliers to determine the dimensions of the largest rectangular parallelepiped which can be inscribed in a hemisphere of radius a.

40. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

41. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is a equilateral.

[**Hint:** Use Heron's formula for the area: $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $s = p/2$ and x, y, z are lengths of the sides.]