Lagrange Multipliers

We want to optimize (*i.e.* find the minimum and maximum value of) a function,

$$f(x,y,z)$$

subject to the constraint

$$g(x,y,z) = k$$

The constraint may be the equation that describes the boundary of a region or it may not be.

<u>Method of Lagrange Multipliers</u>

STEP 1: Solve the following system of equation

$$\begin{aligned} &\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ &g(x,y,z) = k \end{aligned}$$

Where we have to solve these 4 equations simultaneously:

$$\langle f_x, f_y, f_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle \qquad g(x,y,z) = k,$$

or

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x,y,z) &= k \end{aligned}$$

STEP 2: Plug in all solutions, (x,y,z) from the first step into $f(x,y,z)$ and identify the minimum and maximum values, provided they exist.

The constant, λ, is called the **Lagrange Multiplier**.

Example 1: Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution: We want to find the largest volume and so the function that we want to optimize is given by,

$$f(x,y,z) = xyz$$

Next we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides, so the constraint is given by,

$2xy + 2xz + 2yz = 64 \longrightarrow xy + xz + yz = 32$

Thus $g(x, y, z) = xy + xz + yz$.

Here are the four equations that we need to solve.

$$
\begin{array}{lll}
yz = \lambda(y+z) & (f_x = \lambda g_x) & \textbf{(1)} \\
xz = \lambda(x+z) & (f_y = \lambda g_y) & \textbf{(2)} \\
xy = \lambda(x+y) & (f_z = \lambda g_z) & \textbf{(3)} \\
xy + xz + yz = 32 & (g(x,y,z) = 32) & \textbf{(4)}
\end{array}
$$

Hence, $(x, y, z) = (3.266, 3.266, 3.266)$.

Example 2: Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Example 3: Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2$.